RESULTS ON THE AUTOMORPHISM GROUP OF A GRAPH

by

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The undersigned certify that they have read and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled "Results on the Automorphism Group of a Graph", submitted by MYRON GOLDBERG in partial fulfilment of the requirements for the degree of Master of Science.

ABSTRACT

This thesis is devoted to a study of the automorphism groups of graphs in general, and complete oriented graphs (or tournaments) in particular.

A survey of results concerning the automorphism groups of graphs is presented in Chapter I. The remainder of the thesis deals with a particular problem for tournaments, namely, what is the maximum possible order $g(n)$ of the group of a tournament T_n. Actual values of $g(n)$ for $n \leq 27$ are determined in Chapter II. These are used in Chapter III to obtain bounds for $g(n)$ in general. These bounds imply that

$$\sqrt{3} \leq \lim g(n)^{1/n} \leq 2.03 \quad .$$

Finally, it is shown that $g(n)$ is the order of the largest subgroup of odd order in the symmetric group S_n.

The tournaments T_n, $n \leq 6$, and their groups are tabulated in the Appendix.

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A survey of results concerning the automorphism groups of graphs is presented in Chapter I. The remainder of the thesis deals with a particular problem for tournaments, namely, what is the maximum possible order $g(n)$ of the group of a tournament T_n . Actual values of $g(n)$ for $n \leq 27$ are determined in Chapter II. These are used in Chapter III to obtain bounds for $g(n)$ in general. These bounds imply that

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Finally, it is shown that $g(n)$ is the order of the largest subgroup of odd order in the symmetric group S_n .

The tournaments T_n , $n \leq 6$, and their groups are tabulated in the Appendix.

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DEDICATION

This thesis is respectfully dedicated

to the memory of my father,

WILLIAM DAVID GOLDBERG, B.Sc., M.Sc. (Alberta)

(1904 - 1957)

TABLE OF CONTENTS

		Page
	ABSTRACT	(i)
	ACKNOWLEDGMENTS	(ii)
	DEDICATION	(iii)
CHAPTER I:	GRAPHS WITH GIVEN AUTOMORPHISM GROUPS	
§1.1	Introduction	1
§1.2	The Group of Automorphisms of a Graph	2
§1.3	The Cayley Color Graph	7
§1.4	Frucht's Theorem	10
§1.5	Generalizations	16
§1.6	Related Results	23
CHAPTER II:	THE MAXIMUM ORDER OF THE GROUP OF A TOURNAMENT T_n FOR $n \leq 27$	
§2.1	Introduction	26
§2.2	The Case $n = 7$	32
§2.3	Tournaments T_n with Maximum Group for $n = 8,9,10$	33
§2.4	The Case $n = 11$	35
§2.5	Tournaments T_n with Maximum Group for $12 \leq n \leq 22$	38
§2.6	The Case $n = 23$	40
§2.7	Tournaments T_n with Maximum Group for $24 \leq n \leq 26$	50
§2.8	The Case $n = 27$	51
CHAPTER III:	BOUNDS FOR THE MAXIMUM ORDER OF THE GROUP OF A TOURNAMENT	
§3.1	Introduction	53
§3.2	An Upper Bound	53
§3.3	Proof of the Existence of the Limit	55
§3.4	Concluding Remarks	58

BIBLIOGRAPHY 64

APPENDIX 68

CHAPTER I

GRAPHS WITH GIVEN AUTOMORPHISM GROUPS

§1.1 Introduction

A graph G consists of a finite set of vertices (or nodes) some pairs of which are joined by a single edge. We assume that no edge joins a vertex to itself. Two vertices are adjacent if they are joined by an edge; the degree of a vertex is the number of vertices with which it is adjacent. A graph is regular if every vertex has the same degree. The order of a graph G is the number of vertices of G. The complete n-graph has n vertices and $\binom{n}{2}$ edges. A path of length d in G joining vertices p and q is a set of d edges $(p, k_1)\ (k_1, k_2)\ (k_2, k_3)\ \ldots\ (k_{d-1}, q)$. A path of length d joining p to itself is called a cycle of length d. A graph is connected if every pair of points is joined by a path.

Consider an adjacency-preserving permutation α of the vertices of G, so that (a, b) is an edge in G if and only if $(\alpha(a), \alpha(b))$ is an edge in G. It is readily verified that the set of all such permutations forms a group, called the (automorphism) group $\Gamma(G)$ of the graph G. For example, the symmetric group S_n is the group of the complete n-graph. The following question was posed by König [35]: Given a group Γ, does there exist a graph G such that the group of automorphisms of G is isomorphic to Γ? The object of this chapter is to summarize some results on this and related questions.

§1.2 The Group of Automorphisms of a Graph

I. N. Kagno [33] has determined the groups of all graphs of order at most 6 whose vertices have degree at least 3. There is only one such graph of order 4, namely the tetrahedron, and its group is S_4 ; there are 5 such graphs of order 5 and 18 of order 6. All of these graphs have some non-trivial automorphisms. Hemminger [28] has given corresponding results for directed graphs. Kagno [33] gives an example of a graph (see Fig. 1) of order 7 all of whose vertices are of degree at least 3 and whose group is the identity.

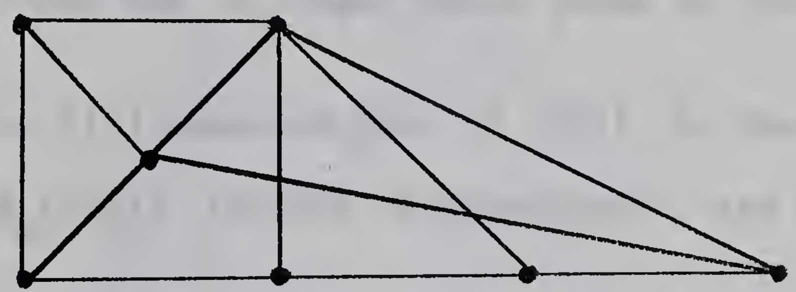


Fig. 1

That the group of this graph is the identity can easily be shown by considering the possible images of a vertex under some automorphism α , and recalling that α is edge-preserving. If it is not insisted that the degrees of the vertices be at least 3 then simpler graphs with the identity group can be found. Examples of such graphs are shown in Fig. 2.

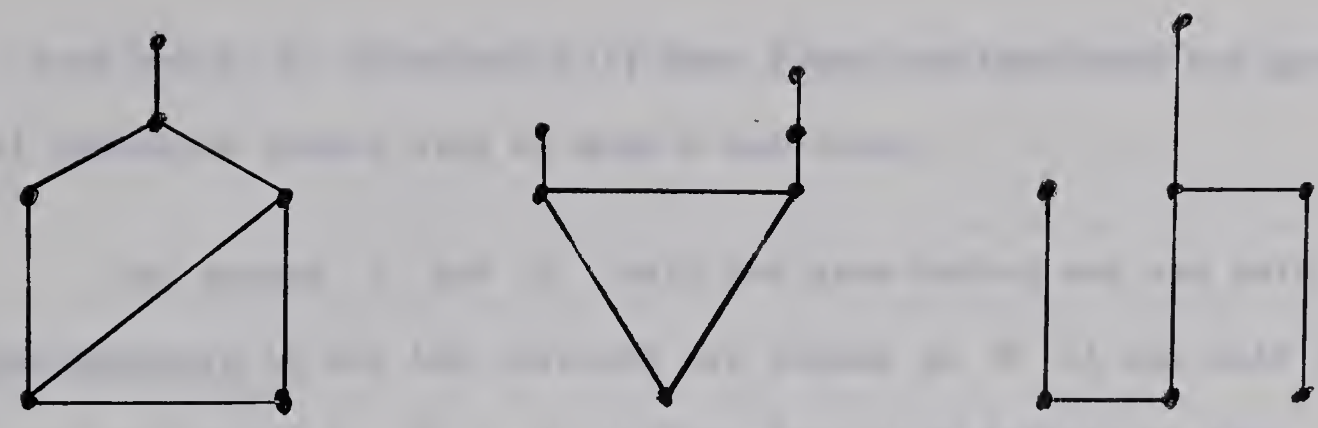


Fig. 2

These examples show that it is easy to construct infinite classes of graphs whose group is the identity. Recently Quintas [40] has determined the extreme values of n and e for which there exist graphs of n vertices and e edges whose group is the identity.

R. Frucht [16] observed that if $\Gamma(G)$ is the group of a graph G, then $S_m[\Gamma(G)]$ (Pólya's "Gruppenkranz", see [38]) is the group of m copies of that graph, considered as a single graph. The elements of $S_m[\Gamma(G)]$ consist of all permutations obtained by permuting the graphs G among themselves and then applying any element of $\Gamma(G)$ to each of them. It follows that in determining the group of a graph G, it suffices to consider the case where G is connected. For, if G is not connected, we may partition G into classes G_i $(i = 1, 2, \ldots, r)$ of isomorphic connected components G_{ij}. Then

$$\Gamma(G) = \Gamma(G_1) \times \Gamma(G_2) \times \ldots \times \Gamma(G_r) ,$$

the direct product of the groups of $G_1, G_2, \ldots, G_r$, where the group $\Gamma(G_i)$ is $S_n[\Gamma(G_{ij})]$.

G. W. Ford and G. E. Uhlenbeck [11] have found and tabulated the groups of all connected graphs with at most 6 vertices.

Two graphs G and G' with the same vertex set are said to be <u>complementary</u> if any two vertices are joined in G if and only if they are not joined in G' . It follows from this definition that $\Gamma(G) = \Gamma(G')$. Kagno [34], in his investigation of the graphical representations of the geometric theorems of Pappus and Desargues, noted that Desargues' graph (see Fig. 3) is the complement of the Petersen graph (see Fig. 4), whose group was already known.

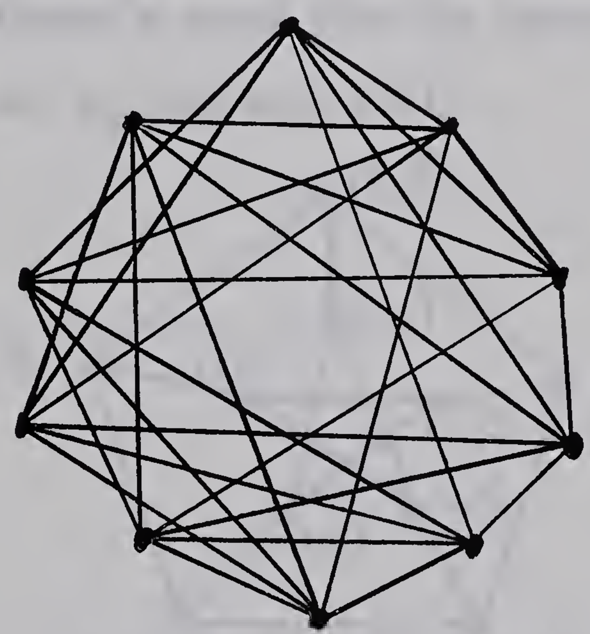


Fig. 3

There does not seem to be any easy way of determining the group of a graph in general. Frequently the difficulty lies not so much in finding automorphisms as in showing that there are none other than those already found.

We remark that Ore [37], Chao [7] and perhaps others have observed that the problem of determining the automorphism group of a graph G is equivalent to determining which permutation matrices commute with the adjacency matrix of G. (The adjacency matrix of a graph G of order n is an $n \times n$ matrix (a_{ij}) where

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge between vertex } i \text{ and vertex } j \\ 0 & \text{otherwise} \end{cases} \Bigg) .$$

As an illustration of the problem of determining the group of a graph, we give Frucht's proof that the group of the Petersen graph is isomorphic to S_5 (see ref. [12]).

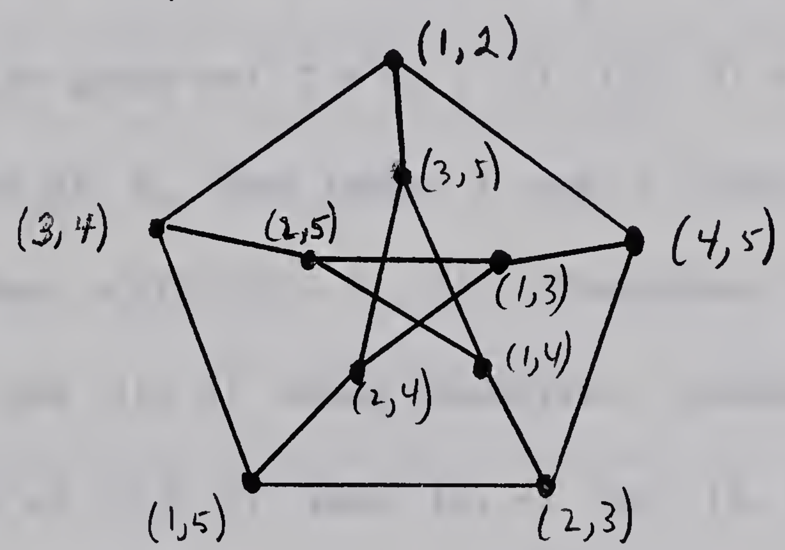


Fig. 4

The nodes are labelled with unordered pairs from the set $\{1,2,3,4,5\}$; two nodes are adjacent if and only if their labels have no numbers in common. It is easily seen that every automorphism $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \end{pmatrix}$ is adjacency-preserving. Different permutations $P = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 \end{pmatrix}$ and

$Q = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \end{pmatrix}$ induce different automorphisms of the graph.

To show this, we may suppose $\alpha_1 \neq \beta_1$. If P and Q induce the same automorphism, then $Q(1, 2) = (\beta_1, \beta_2) = (\alpha_1, \alpha_2) = P(1, 2)$ so that $\beta_1 = \alpha_2$ and $\beta_2 = \alpha_1$. Also $Q(1, 3) = (\beta_1, \beta_3) = (\alpha_1, \alpha_3) = P(1, 3)$ so that $\beta_1 = \alpha_3$ and $\beta_3 = \alpha_1$. But β_1 cannot be equal to both α_2 and α_3, hence P and Q must induce different automorphisms. It follows that S_5 is a subgroup of the group of the Petersen graph.

The proof will be complete if we can show that there are no automorphisms other than those induced by S_5. Suppose T is an automorphism of the graph and $T \notin S_5$. If $T(1, 2) = (\gamma, \lambda)$, let A_1 be any element of S_5 that takes γ and λ into 1 and 2 respectively. Then $A_1T(1, 2) = (1, 2)$. Therefore, A_1T permutes $(3, 4)$, $(3, 5)$ and $(4, 5)$ among themselves. Suppose A_1T takes $(3, 4)$ into (ρ, σ), $(3, 5)$ into (ρ, τ) and $(4, 5)$ into (σ, τ). Let $A_2^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & \rho & \sigma & \tau \end{pmatrix}$. Then A_2A_1T leaves fixed $(1, 2)$, $(3, 4)$, $(3, 5)$ and $(4, 5)$, and either interchanges $(1, 5)$ and $(2, 5)$ or leaves them fixed. In the first case let $A_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \end{pmatrix}$ and in the second case let $A_3 = I$. In either case we find that $A_3A_2A_1T$ leaves fixed $(1, 2)$, $(3, 4)$, $(3, 5)$, $(4, 5)$, $(1, 5)$, $(2, 5)$ and, consequently, everything else. It follows that $A_3A_2A_1T = I$ so that $T = (A_3A_2A_1)^{-1} \in S_5$. This completes the proof.

§1.3 The Cayley Color Graph

We now introduce a graphical representation of groups due to Cayley [4,5]. We assume in what follows that all groups under discussion are non-trivial (see the remark after Fig. 2). With each element of a group Γ of order h we associate a vertex of a graph G ; with each element g_j of Γ we associate a set of directed edges in G which will be said to have color j . There exists an edge of color j oriented from vertex p to vertex q if and only if

$$pg_j = q \quad (j = 1, 2, \ldots, h) \quad .$$

(The same symbol is used for the vertex and the corresponding group element.) Hence at each vertex there are two edges of each color - one directed towards the vertex and one directed away. In particular, there will be a loop at each vertex with the color of the identity element of Γ . Any product of elements in Γ that yields the identity corresponds to a cycle in G . The connected graph G just described is called <u>the complete Cayley color graph</u> of the group Γ .

For our purposes, it will be sufficient to consider the "reduced" color graph G' of a group. This differs from the graph described above in that all edges are discarded except those that have the colors of some minimal set of generators of Γ . Since each element in a group can be expressed as the product of generators it follows that G' is still connected. We note that G' has no loops, since the identity is not among the generators. Furthermore, no two vertices in G' will be joined by two edges of different colors oppositely oriented,

since this would correspond to two elements which are inverses of each other, and such elements are not in a minimal set of generators. In particular then, any cyclic group of order n can be represented graphically as a cycle of length n.

Figure 5 shows the color graph of the group S_3 of symmetries of the triangle.

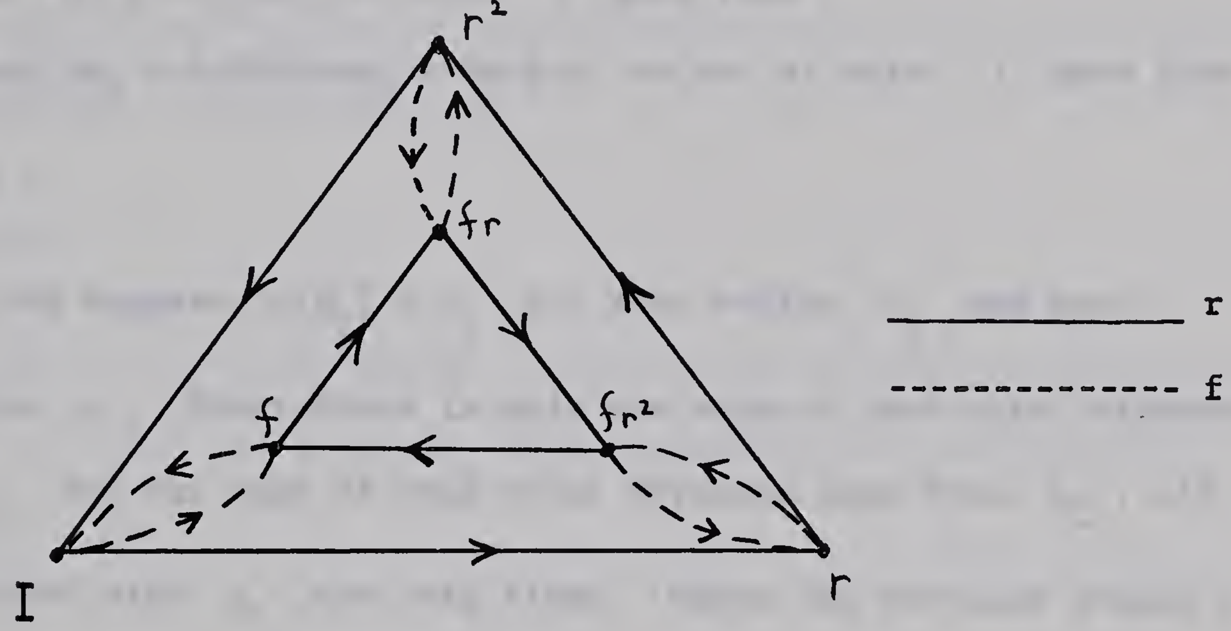


Fig. 5

The group S_3 is generated by a rotation r and a reflection f.

The most important property of the Cayley color graph, as far as we are concerned, is given by the following result.

<u>Theorem 1.3.1</u>

The group of automorphisms that preserve the color and orientation of the edges of the Cayley color graph of a group Γ is isomorphic to Γ.

Proof

We shall prove this by showing that such automorphisms of the Cayley color graph are precisely those that are obtained by left multiplication of the vertices with group elements.

We observe first that left multiplications are indeed color preserving, since an edge of color j goes from p to $q \Longleftrightarrow pg_j = q \Longleftrightarrow \alpha pg_j = \alpha q \Longleftrightarrow$ an arc of color j goes from αp to αq .

Now suppose $\alpha(g_i) = g_i$ for some vertex g_i and some automorphism α . Since there is only one edge of each color directed towards g_i and one edge of each color directed away from g_i , all edges incident with g_i are left fixed. Hence the vertices joined to g_i are also left fixed by α . By repeating this argument for each node it follows from the fact that G is connected that any automorphism fixing a vertex must be the identity.

Suppose that $\alpha(p) = q$ and $\beta(p) = q$. Then $\beta^{-1}\alpha(p) = p$ so that $\beta^{-1}\alpha = e$, that is, $\alpha = \beta$. This shows that if any two automorphisms "agree" on a vertex, they are identical. Hence there can be at most one automorphism α such that $\alpha(p) = q$ for any two vertices p and q .

Let $L_g(x) = gx$ denote the left multiplication of $x \in G$ by $g \in \Gamma$. If φ denotes any color-preserving automorphism of the

color graph let $\varphi(q) = q'$. But $q' = L_{q'q^{-1}}(q)$. It follows from the above remarks that $\varphi = L_{q'q^{-1}}$ and hence every color preserving automorphism is a left multiplication. This completes the proof of the theorem.

§1.4 Frucht's Theorem

The following theorem due to Frucht [13] answers the question raised by König as stated in §1.1.

Theorem 1.4.1

For every finite group Γ there exists a finite undirected graph G such that the group of automorphisms of G is abstractly isomorphic to Γ .

Proof

The required graph G is constructed by transforming the reduced color graph $C(\Gamma)$ of Γ into an undirected graph all of whose edges have the same color, or equivalently, no color at all, without changing its group. This is accomplished by replacing the colored edges of $C(\Gamma)$ by members of a family of non-isomorphic connected graphs each of whose groups is the identity. We will use the family of graphs T_i illustrated in Figure 6 .

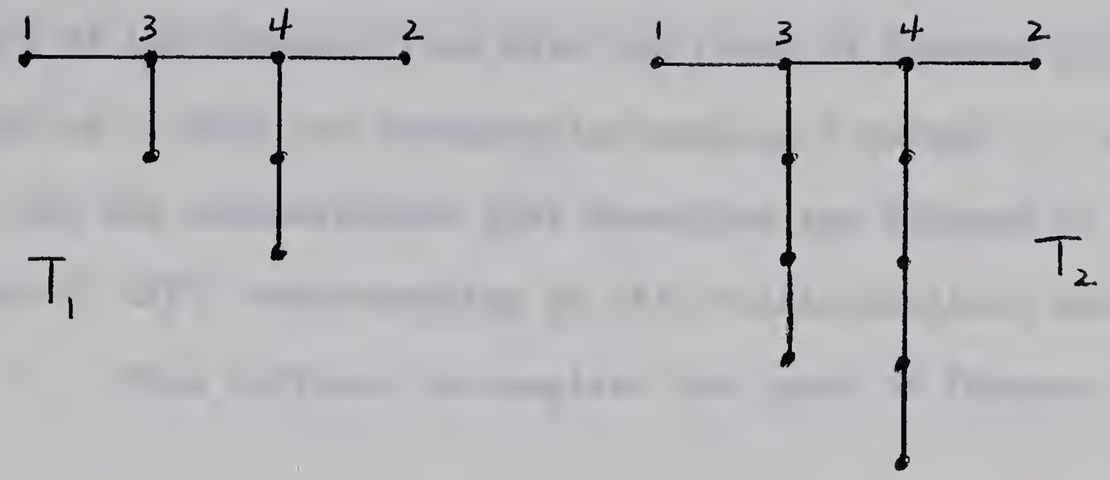


Fig. 6

Each graph T_i consists of the edges $(1, 3)$, $(3, 4)$ and $(4, 2)$ and from vertices 3 and 4 there are "tails" of length $2i - 1$ and $2i$ respectively for $i = 1, 2 \ldots$. In the color graph $C(\Gamma)$ replace each directed edge $\overrightarrow{pq}$ of color j by T_j in such a way that p corresponds to vertex 1 of T_j and q corresponds to vertex 2.

To show that the group of the graph G just described is isomorphic to Γ we first observe that any automorphism α fixing a vertex p must be the identity; for, the edges from p are either of the type $(1, 3)$ or $(2, 4)$ depending on the direction of the corresponding edge in $C(\Gamma)$. But $\alpha(p) = p$ implies that the "neighbors" 3 and 4 of p are fixed because their tails have different lengths. Hence the graphs T_i at p are fixed and also the vertex 2 corresponding to a vertex q. Continuing in this way we see that α fixes all vertices of G. Secondly, we observe that a vertex p in G corresponding to a vertex p' in $C(\Gamma)$ can only be mapped into another such vertex, never into a vertex on a tail, since its degree and the degrees of its adjacent vertices would not be preserved otherwise. Thirdly, it follows from

the first part of the argument (see also the proof of Theorem 1.3.1) that there can be at most one automorphism mapping a vertex p into a vertex q. But the automorphisms just described are induced by the automorphisms of $C(\Gamma)$ corresponding to left multiplications with elements of Γ. This suffices to complete the proof of Theorem 1.4.1.

We mention the following point concerning isomorphisms of permutation groups. In our discussion, we consider two permutation groups as isomorphic only in the "abstract" or "simply isomorphic" sense as opposed to the stronger "permutationally-isomorphic" sense. Specifically, if R and G are permutation groups with respective object sets X and Y, we say that R and G are permutationally-isomorphic if there exists a one-to-one mapping $f : X \to Y$ and a group isomorphism $g : R \to G$ such that $f(\alpha x) = (g\alpha) f(x)$ for all $x \in X$, $\alpha \in R$. For example, the two cyclic groups of order 6 generated by $(1\ 2\ 3\ 4\ 5\ 6)$ and $(1\ 2\ 3)(4\ 5)$ respectively are simply isomorphic but not permutationally-isomorphic. Theorem 1.4.1 is true for simple isomorphisms, but there are some groups Γ for which there does not exist a graph G whose automorphism group is permutationally isomorphic to Γ; for example, Kagno [33] has shown that cyclic or alternating groups are not realizable as groups of graphs in this sense. For graphs of order at most 6, he gives all the cases for which a solution exists. In general, the problem of determining when a group can be realized as the group of a graph in the sense of permutation isomorphism remains unsolved and seems to be difficult.

Figure 7 gives an illustration of the graph G obtained from the color graph in Figure 5.

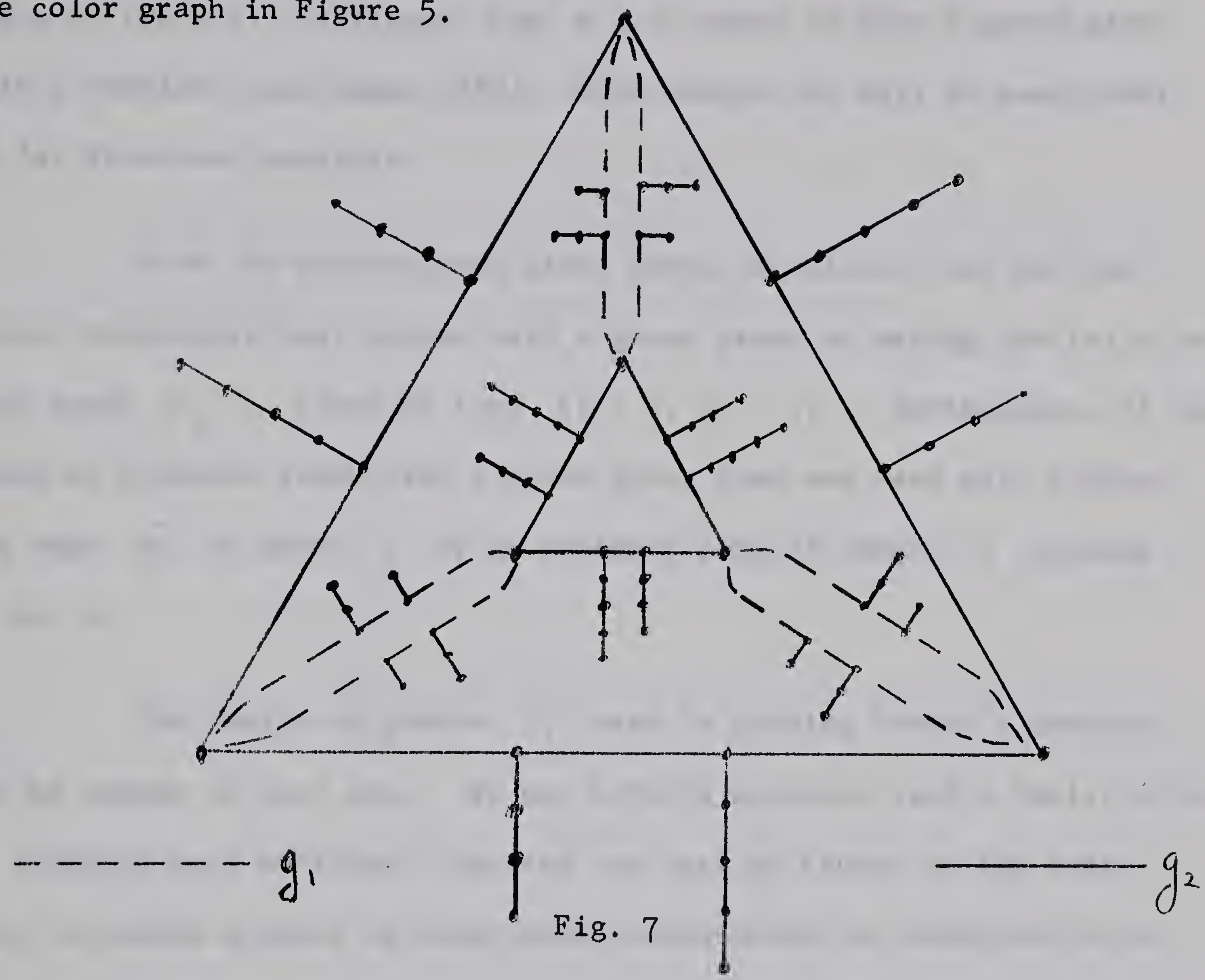


Fig. 7

In the proof of Theorem 1.4.1 it was shown that i) any two automorphisms which agree at a vertex are identical, and ii) the graph constructed having the required properties was in some sense a generalized Cayley color graph. We remark that Sabidussi [42] has proved that i) implies ii). Chao [6] has used this result to prove that there is no graph with n vertices whose automorphism group is transitive and abelian if $n > 2$.

Generally, then, we have seen that the problem of whether or not there exists a graph whose automorphism group is simply isomorphic to a given group of order $n > 1$ and generated by h of its elements

always has a solution; the graph constructed in the theorem has $n(h+1)(2h+1)$ vertices. For $n = 1$ there is such a graph with only 6 vertices (see Kagno [33]). This result, as will be seen later, is far from best possible.

From the construction given above it follows that one can obtain infinitely many graphs with a given group by making the tails on each graph T_i t times as long $(t = 1, 2, \ldots)$. Furthermore, if one wants an oriented graph with a given group then one need only replace the edge $\overrightarrow{pq}$ of color j by an oriented path of length j joining p to q.

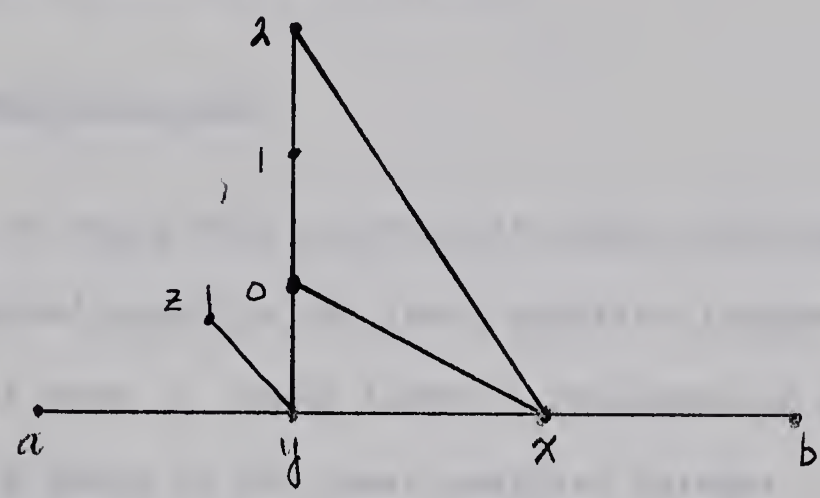
The family of graphs T_i used in proving Frucht's theorem may be chosen in many ways. We now briefly describe such a family which is slightly more efficient than the one used by Frucht in the sense that it yields a graph of lower order whose group is isomorphic to a given group.

If the group Γ of order n is generated by h of its elements, choose t such that $2^{t-1} \leq h < 2^t$. If

$i = \sum_{j=0}^{t-1} b_j 2^j$, where $b_j = 0$ or 1, then let the graph T_i^*

consist of edges (a,y), (z, y), (y, x), (x, b) and from vertex y there is a path of length t whose vertices are labelled $0, 1, 2, \ldots, t-1$. There exists an edge joining x to j $(0 \leq j \leq t-1)$ if and only if $b_j = 1$ (an example is shown in Fig. 8). Thus each

T_i^* $(i = 1, 2, \ldots, h)$ introduces $t + 3$ new vertices to the color graph for each edge $\overrightarrow{pq}$ of color i, where, as before, the vertex p is identified with the vertex a, and q with b. There were n points and nh edges in the color graph to start with. It follows that no more than $n + nh([\log_2 h] + 4)$ vertices are required as opposed to roughly $2nh^2$ by Frucht's construction.



$h = 6$

$i = 5 = (1\ 0\ 1)$

Fig. 8

Later Frucht [15] himself gave a more efficient construction. He proved that for a given group Γ of order $n > 2$ and generated by h of its elements there exists a cubical graph (i.e. a graph all of whose vertices have degree 3) with $2n(h + 2)$ vertices whose group is isomorphic to Γ; for $n = 1$ or 2 the graphs have 12 or 10 vertices respectively. Frucht then modified his new construction to give yet another solution to König's problem but without the extra condition that the graph be cubical. He proved that instead of $n(h + 1)(2h + 1)$ vertices all that are needed are $2hn$ vertices for non-cyclic groups and $3n$ vertices for cyclic groups of order $n > 3$ and 10 vertices if $n = 3$.

In his arguments, Frucht defines a graph by means of a quadratic form; the group of the graph is the group of permutations of the variables which leave the quadratic form fixed.

Theorem 1.4.1 has been extended to include infinite groups by Sabidussi [45] and also by de Groot [20], who obtains an analogous result for directed graphs as a corollary.

§1.5 Generalizations

We begin this section with some definitions. The <u>connectivity</u> of a connected graph is the least positive integer n such that the deletion of some n nodes leaves a disconnected graph. The <u>chromatic number</u> of a graph is the least positive integer n such that the graph can be divided into n mutually disjoint sets so that no two vertices of the same set are joined by an edge. An n-chromatic graph is called <u>critical</u> if it has no proper n-chromatic subgraphs. Two graphs are <u>homeomorphic</u> if they are isomorphic or if they can be made isomorphic by repeatedly subdividing edges in two by inserting additional vertices in one or both graphs. (An example of two homeomorphic graphs is given in Fig. 9). The <u>composition</u> $G_1 \circ G_2$ of graphs G_1 and G_2 is the graph G formed by replacing each point of G_1 by a copy of G_2 and each edge of G_1 by edges joining all pairs of vertices in corresponding copies of G_2. A graph is <u>fixed-point-free</u> if no vertex is fixed under all automorphisms of the graph.

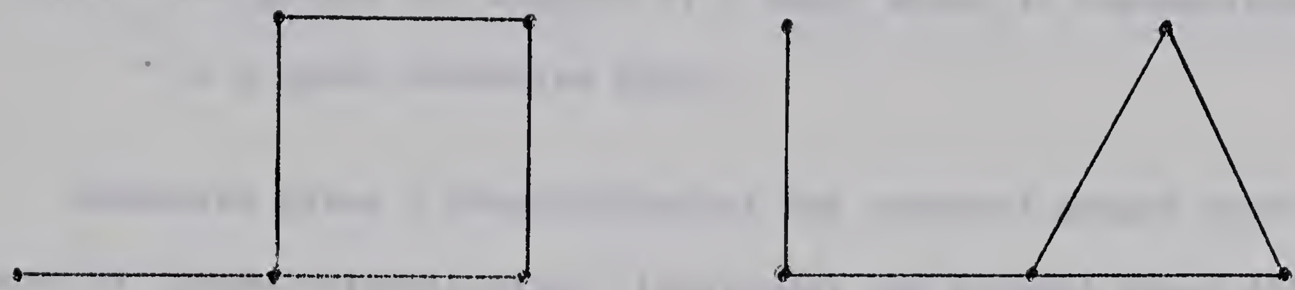


Fig. 9

Izbicki [29] and Sabidussi [41] showed that the fact that a graph has a prescribed group does not severely restrict its other properties. Izbicki's theorem is the following.

Theorem 1.5.1

Given any finite group Γ, and integers n, k and c such that $3 \leq n \leq 5$, $2 \leq k \leq n$, $1 \leq c \leq n$ and $(k, c) \neq (2, 1)$ there exist infinitely many non-isomorphic graphs regular of degree n, having chromatic number k, connectivity c, and whose group of automorphisms is isomorphic to Γ.

The following result is due to Sabidussi.

Theorem 1.5.2

Given any finite group Γ, for each of the following properties there exist infinitely many non-isomorphic, finite, connected, fixed-point-free graphs whose group of automorphisms is isomorphic to Γ and which possess that property:

i) The graphs are of connectivity n.

ii) The graphs are of chromatic number $n \geq 2$.

iii) The graphs are regular of degree $n \geq 3$.

iv) The graphs are spanned by a graph which is homeomorphic to a given connected graph.

Sabidussi gives a construction of the required graphs based on his method of "graph multiplication" (Sabidussi has defined three different types of graph products. For the definitions and properties of these products, the reader is referred to reference [46]). His argument is based on (a) finding one graph G with the required group, (b) finding a family of graphs each of whose group is the identity and with certain graph-theoretical properties, (c) taking "products" of graphs in (b) with those in (a).

Recently Chao [7] has investigated the automorphism groups of graphs from the viewpoint of the theory of permutation groups. For a graph G and a subgraph G^* of G, a sufficient condition is given for $\Gamma(G^*)$ to be a normal subgroup of $\Gamma(G)$. It is proved by construction that for any nontrivial finite group Γ there exists a graph G and a subgraph G^* such that the quotient group $\Gamma(G)/\Gamma(G^*)$ is isomorphic to Γ. An algorithm is given for obtaining all graphs G on n vertices whose group contains a given transitive group of degree n.

It is clear that among all the graphs whose groups are isomorphic with a given group Γ there must be one graph having a minimal number $\alpha(\Gamma)$ of vertices. Sabidussi [43] proved the following result.

Theorem 1.5.3

If the group Γ of order n is generated by h of its elements, then

$$\alpha(\Gamma) = O(n \log h) \ .$$

He also observed that since $h = O(\log n)$ then $\alpha(\Gamma) = O(n \log \log n)$. It seems hopeless to obtain a best possible estimate for $\alpha(\Gamma)$. Explicit values of $\alpha(C_m)$ where C_m is the cyclic group of order m are also given. (See also Harary and Palmer [24]).

A connected graph is said to be s-regular if there exists an automorphism which maps any directed arc of length s into any other such arc. Tutte [49] and Coxeter [8] discovered which cubical graphs were s-regular. Tutte found that there are no s-regular graphs for $s > 5$, and gave examples for $s = 2, 3, 4, 5$. The Petersen graph (see Fig. 4) is an example of a 3-regular graph. Frucht [19] constructed a one-regular cubical graph. It has 432 vertices.

Results have been obtained regarding the groups of graphs upon which certain operations have been defined. Sabidussi [44], in order to correct a theorem of Harary [23] on the group of the composition of two graphs, made the following definitions. Let $G_1 \circ G_2$ be the composition of two graphs G_1 and G_2 with groups Γ_1 and Γ_2 respectively. Let V_1 be the vertex set of G_1 and define a neighborhood $N(v)$ of $v \in V_1$ as the set of vertices adjacent to v in G_1. Define the following equivalence relations R and S on V_1:

a) uRv if $N(u) = N(v)$

b) uSv if $N(u) \cup \{u\} = N(v) \cup \{v\}$

Sabidussi's result is the following.

Theorem 1.5.4

$\Gamma(G_1 \circ G_2)$ is isomorphic to $\Gamma_1 \circ \Gamma_2$ if and only if

a) G_2 is connected if R is not the identity, and b) the complement of G_2 is connected if S is not the identity.

In the above theorem, $\Gamma_1 \circ \Gamma_2$ is the wreath product of Γ_1 by Γ_2 defined as follows: $\Gamma_1 \circ \Gamma_2$ is the group of all permutations f on $V(G_1) \times V(G_2)$ (the Cartesian product of the vertex sets of G_1 and G_2) of the following kind:

$$f(a, b) = (a\gamma_b, bg) \quad , \quad a \in V(G_1) \ , \quad b \in V(G_2)$$

where for each $b \in V(G_2)$, γ_b is a permutation of Γ_1 on $V(G_1)$, but for different b's the choices of the permutations γ_b are independent. The permutation g is a permutation of Γ_2 on $V(G_2)$. This is equivalent to Pólya's Gruppenkranz $\Gamma_2[\Gamma_1]$, a special case of which was mentioned earlier.

The example given in Fig. 10 shows the necessity of the hypotheses of Sabidussi's theorem. It is easy to see that $|\Gamma(X \circ Y)| \neq |\Gamma(X) \circ \Gamma(Y)|$.

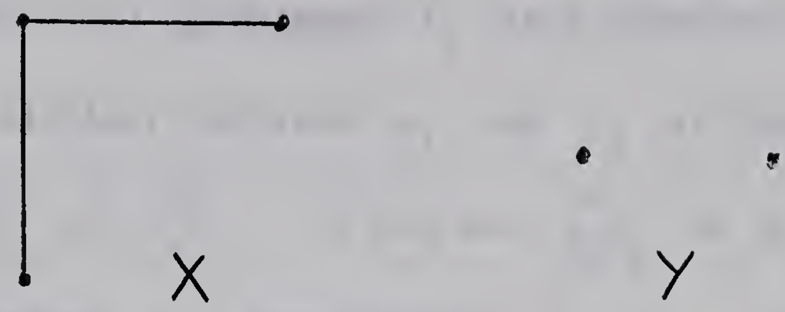


Fig. 10

Izbicki [32] has also investigated the relationships between the group of a graph G and the group of its image under certain operators $f : G \to f(G)$. Among the nine operators considered are complement, dual, and removal of loops.

The <u>line graph</u> of a graph G is the graph whose vertices correspond to the edges of G and in which two vertices are adjacent if and only if the corresponding edges of G have a vertex in common. The line graph of G has been called the <u>interchange graph</u> of G by Ore and the <u>derivative</u> of G by Sabidussi. Sabidussi [47] proved that if G is a connected graph, not isomorphic to any of four particular graphs, then G and its line graph have isomorphic automorphism groups. The four exceptional graphs are the complete 2-graph, the complete 4-graph, and the graphs P and Q shown below.

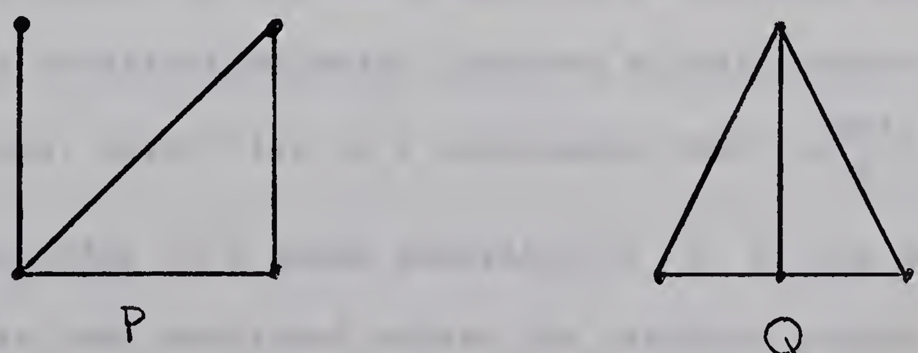


Fig. 11

A (round-robin) <u>tournament</u> T_n is a graph on n vertices such that each pair of distinct vertices p_i and p_j is joined by one of the oriented arcs $\overrightarrow{p_i p_j}$ or $\overrightarrow{p_j p_i}$. If the arc $\overrightarrow{p_i p_j}$ is in T_n, then we say that p_i <u>dominates</u> p_j, symbolically $p_i \rightarrow p_j$. Thus, a tournament is a complete oriented graph. A tournament T_n is said to be <u>reducible</u> if it is possible to partition its nodes into two disjoint and exhaustive subsets A and B such that each vertex of A dominates each vertex of B. A tournament is <u>irreducible</u> (strongly connected) if it is not reducible. It is clear that the set of all dominance-preserving permutations of the vertices of T_n form a group, the automorphism group $G(T_n)$ of T_n. The following result is due to Moon [36].

<u>Theorem 1.5.5</u>

There exists a tournament whose automorphism group is isomorphic to a given group Γ if and only if Γ is of odd order.

Moon showed that there are infinitely many such tournaments and all of them are strongly connected. The necessity of Theorem 1.5.5 follows upon recalling that any group of even order has at least one involutory element $\alpha \neq e$, which, as a permutation, is clearly not dominance-preserving. Moon's construction, which involves a modification of the color graph of the group, gives rise to a tournament with $n\binom{h+1}{2} + n$ vertices, where n is the order of a group generated by h of its elements. Similarly to what has been mentioned before for ordinary graphs, some particular groups will have tournaments with far fewer vertices. For example,

if the given group Γ is abelian, then it can be expressed as the direct product of cyclic groups, each of which is isomorphic to the group of a tournament with the same number of vertices as the order of the group. Hence, if $\Gamma \cong C_m \times C_n$ let A and B be tournaments with groups C_m and C_n respectively; the group of the tournament obtained by having every vertex in A dominate every vertex in B is isomorphic to $C_m \times C_n$. However, these tournaments are not strongly connected unless Γ itself is cyclic.

In the remaining two chapters, we take a closer look at certain aspects of the automorphism group of a tournament. The tournaments with up to six vertices and their groups are displayed in the Appendix.

§1.6 Related Results

Among the investigations which have been carried out regarding extensions of Frucht's theorem, some of the earlier results concerned the automorphism groups of partially ordered systems. In 1946, G. Birkhoff [2] proved the following

<u>Theorem 1.6.1</u>

Each abstract group Γ of order g is isomorphic to the group of automorphisms of some partially ordered system X with $g^2 + g$ elements.

In 1948 Frucht [14] showed that the number of elements in X can be reduced to g^2.

The following result is due to Frucht [17].

Theorem 1.6.2

Each abstract group Γ of order g generated by n of its elements is isomorphic to the group of automorphisms of some partially ordered system X with $(n + 2)g$ elements.

If $n > 2$ then $(n + 2)g$ can be replaced by $(m + 1)g$ where m is the least integer such that $m \geq \frac{1}{2}[1 + (8n + 1)^{1/2}]$.

In 1950, Frucht [18] proved the following analogous results for lattices.

Theorem 1.6.3

Every group Γ of order g generated by n of its elements is isomorphic to the group of automorphisms of a lattice with at most $5g(n + 2) + 2$ elements.

Theorem 1.6.4

For every graph G with n vertices and e edges and no isolated vertices, there exists a lattice with $e + n + 2$ elements whose automorphism group is isomorphic to that of G .

We remarked earlier that de Groot [20] gave a proof of Frucht's theorem for arbitrary groups. More generally, he proved that for every group Γ there exists a complete, connected, locally connected metric space of any positive dimension whose autohomeomorphism group (i.e. the

group of all topological transformations of a topological space to itself) is isomorphic to Γ . It follows from this that every group is the automorphism group of some ring, in fact of infinitely many commutative rings with an identity.

With regard to the automorphism groups of trees (i.e. connected graphs with no cycles) we remark that G. Prins [39] has characterized those groups for which there exist trees whose automorphism groups are permutationally isomorphic to the given group.

An <u>endomorphism</u> of a graph G is a function f from $V(G)$ to $V(G)$ such that if (x,y) is an edge in G then $(f(x), f(y))$ is an edge in G . The set of endomorphisms of a graph G forms a semigroup with unit. Recently, Hedrlín and Pultr [27] proved the following result.

<u>Theorem 1.6.5</u>

Let S be a semigroup with unit whose cardinality is less than the first inaccessible cardinal. Then there exists a graph G whose semigroup of endomorphisms is isomorphic to S .

The same authors (see [25] and [26]) also obtained analogous results with respect to directed graphs.

CHAPTER II

THE MAXIMUM ORDER OF THE GROUP OF A TOURNAMENT T_n FOR $n \leq 27$.

§2.1 Introduction

If $g(T_n)$ denotes the order of the group $G(T_n)$ of a tournament T_n , let $g(n)$ denote the maximum of $g(T_n)$ taken over all tournaments T_n . The exact values of $g(n)$ for $n \leq 27$ are given in Table 1. In Chapter III we will use these values to obtain bounds for $g(n)$ by induction.

The values of $g(n)$ for $n \leq 6$ may be obtained from the Appendix. There does not seem to be any convenient method of determining $g(n)$ in general. Although the ideas involved are fairly straightforward, their repeated application in the large number of special cases becomes rather tedious. We give the details of the argument only for $n = 7, 11, 23,$ and 27 since no new ideas are used in the remaining cases.

Some preliminary observations may be made to facilitate the examination of the tournaments T_n , the number of which increases rapidly with n .

Consider any node p of an arbitrary tournament T_n . Let d denote the number of different nodes in the set

$$D = \{ \alpha(p) : \alpha \in G(T_n) \} .$$

n	g(n)	$g(n)^{\frac{1}{n-1}}$
1	1	-
2	1	1
3	3	$\sqrt{3}$ = 1.7321
4	3	1.442
5	5	1.495
6	9	1.552
7	21	1.662
8	21	1.545
9	81	$\sqrt{3}$ = 1.732
10	81	1.629
11	81	1.552
12	243	1.647
13	243	1.581
14	441	1.597
15	1215	1.661
16	1701	1.643
17	1701	1.592
18	6561	1.677
19	6561	1.629
20	6561	1.588
21	45,927	1.710
22	45,927	1.667
23	45,927	1.629
24	137,781	1.673
25	137,781	1.637
26	229,635	1.639
27	1,594,323	$\sqrt{3}$ = 1.732

TABLE I

We call D an orbit of length d, or simply a d-orbit. If $d = n$ we say that T_n is symmetric. It is clear that D determines a subtournament T_d in which every node is similar to every other node under the action of $G(T_n)$. We do not distinguish between an orbit and the (sub)tournament determined by the nodes of the orbit. Since there are $\frac{1}{2}d(d-1)$ edges in T_d, it must be that each node of T_d dominates $\frac{1}{2}(d-1)$ other nodes of T_d (the number of nodes dominated by a given node x is called the score of x). It follows that d must be odd.

We will say that two disjoint subsets of nodes of a tournament are independent if the edges joining nodes in different subsets all have the same orientation; if the subsets are orbits this means that the nodes of one orbit may be permuted independently of those of the other orbit. If every node of orbit A dominates every node of orbit B, then we say that orbit A dominates orbit B and write $A \to B$.

Let T_{a+b} be a tournament composed of two disjoint orbits T_a and T_b. If T_a and T_b are independent then

$$G(T_{a+b}) = G(T_a) \times G(T_b) .$$

Clearly, if the two orbits are not independent, then $G(T_{a+b})$ must be a subgroup of $G(T_a) \times G(T_b)$ and hence $g(T_{a+b})$ must divide $g(T_a)g(T_b)$. Since this observation will be used later, we state it formally as a lemma.

<u>Lemma 2.1.1</u>

Let T_a and T_b be given tournaments and let T_{a+b} be any tournament obtained by joining every node of T_a to every node of T_b. Then

$$G\,(T_{a+b}) \leq G\,(T_a) \times G\,(T_b)$$

and

$$g(T_{a+b}) \text{ divides } g(T_a)g(T_b) \quad .$$

Consider a given tournament T_n and some node $p \in T_n$, and let the orbit D of p be of length d. If T_d and T_{n-d} denote the subtournaments determined by the nodes in D and the nodes not in D, then it is clear that

$$(2.1.1) \qquad g(T_n) \leq g(T_d)g(T_{n-d}) \leq g(d)g(n-d) \quad .$$

Equality holds in inequality (2.1.1) if (but not necessarily only if) T_d and T_{n-d} are independent.

(A more general form of (2.1.1) may be obtained. Since $G(T_n)$ partitions the nodes of T_n into disjoint orbits T_a, T_b, ..., T_r, then

$$(2.1.2) \qquad g(T_n) \leq g(T_a)g(T_b) \ldots g(T_r) \leq g(a)g(b) \ldots g(r)$$

where $a + b + \ldots + r = n$,

with equality holding if and only if the orbits are pairwise independent.)

If n is even, then necessarily $d < n$, and it follows from (2.1.1) that

$$(2.1.3) \qquad g(n) = \max \{ g(d)g(n-d) \}, \quad d = 1,3,5,\ldots,n-1 .$$

(Strictly speaking, one need only consider d such that $2d \leq n$.)

When n is odd, there is the possibility that $d = n$ in which case (2.1.1) is of no use. A different argument must be used in this case. We appeal to a result in group theory (see Wielandt [51] p.5).

Lemma 2.1.2

If G is a permutation group acting on a set A, let G_α denote the subgroup of G fixing $\alpha \in A$ and α^G denote the orbit of α under the action of G. Then

$$|G| = |G_\alpha| \cdot |\alpha^G| .$$

Suppose now that $d = n$ for some tournament T_n. We have observed that this implies

(a) every node of T_n is similar to every other node under $G(T_n)$,

(b) every node of T_n dominates $\frac{1}{2}(n-1)$ other nodes of T_n, and

(c) n is odd.

Consider the subgroup H of automorphisms α of $G(T_n)$ such that $\alpha(p) = p$. It follows from Lemma 2.1.2 that if $d = n$, then

$$(2.1.4) \qquad g(T_n) = n|H| .$$

No element of H can transform one of the $\frac{1}{2}(n-1)$ nodes that dominate p into one of the $\frac{1}{2}(n-1)$ nodes dominated by p since p is fixed under H . Hence,

$$(2.1.5) \qquad g(n) \leq n\left[g(\tfrac{1}{2}(n-1))\right]^2 .$$

In view of (2.1.3) and the above remarks, it follws that in determining the exact values of $g(n)$ the only tournaments that need to be examined individually are the symmetric ones, that is, those with an odd number of nodes in which all the nodes are similar to each other. The group of such a tournament, however, may not have order $g(n)$. Hence, for n odd,

$$(2.1.6) \qquad g(n) \leq \max\left\{\left[\max\,(g(d)g(n-d)\right] , \; n\,g^2(\tfrac{n-1}{2})\right\} .$$

We may summarize the general method to be used in determining $g(n)$ as follows. To shown that $g(n) = k$, we first show that

$$\max\left\{g(d)g(n-d)\right\} \leq k$$

for all relevant values of d . When n is odd, we must also show, using (2.1.4), that

$$g(T_n) \leq k$$

for all symmetric tournaments T_n . Finally, we exhibit a tournament T_n whose group has order k .

§2.2 The Case $n = 7$.

We prove in this section that $g(7) = 21$. Straightforward calculations using the information in the Appendix show that

$$\max_d \left\{ g(d)g(7-d) \right\} = 9 .$$

By the remarks in §2.1 it remains to consider the case where T_7 is symmetric. It follows from (2.1.4) that

$$g(T_7) \leq 7 \cdot g(3) \cdot g(3) = 63 .$$

We consider as fixed an arbitrary node p and denote by A the set of nodes dominated by p and by B the set of nodes which dominate p . Assume that A and B each form 3-orbits.

A group of order 63 could be realized only if these 3-orbits were independent, in which case the edges joining nodes of the two 3-orbits all would have the same orientation. This situation is impossible since each node of the dominating 3-orbit would have score at least 4. It follows that there cannot exist two independent 3-orbits and hence if there are two 3-orbits they must permute in tandem. Therefore,

(2.2.1) $$g(T_7) \leq 7 \cdot g(3) = 21 .$$

This inequality holds if there exists only one 3-orbit, and if there are no non-trivial orbits under the action of the subgroup fixing p then $g(T_7) \leq 7$.

Equality holds in inequality (2.2.1) for the symmetric tournament T_7 in which the node x_i dominates the node x_j if and only if $j - i$ is a quadratic residue modulo 7 (see Fig. 12). It is not difficult to see that the group of this tournament is the non-abelian group of order 21 generated by $T = (1)(235)(764)$ and $S = (1234567)$. (This is the smallest non-abelian group of odd order. See Coxeter and Moser [9] p.134). This suffices to prove that $g(7) = 21$.

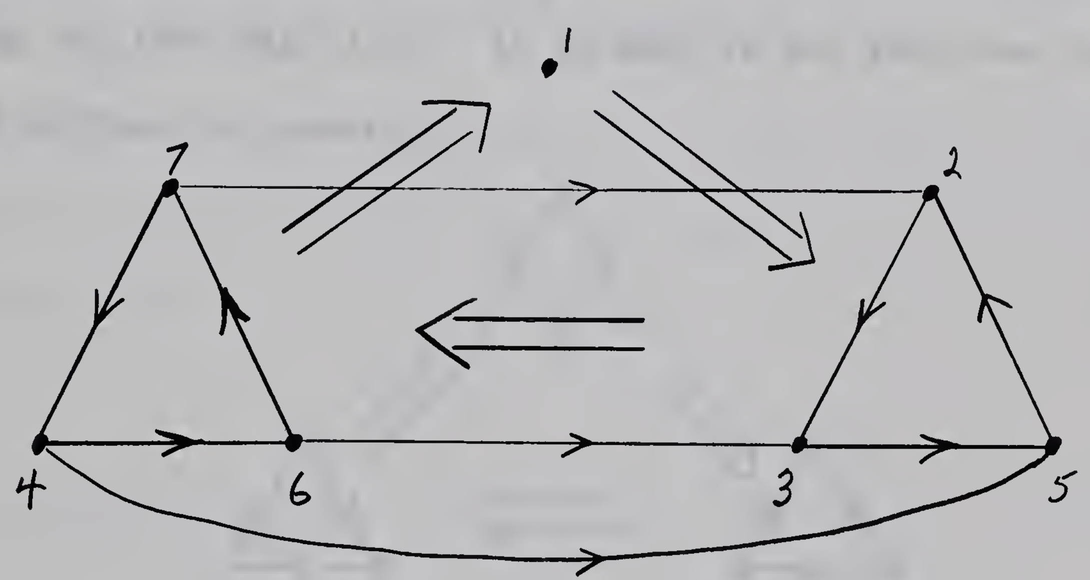


Fig. 12

§2.3 Tournaments T_n with Maximum Group for $n = 8,9,10$.

In this section we construct a tournament T_n whose group is of order $g(n)$ for $n = 8,9,10$.

For $n = 8$, (2.1.3) shows that $g(n) = 21$. A tournament T_8 whose groups is of order 21 is easily obtained by adding an independent node (i.e. independent in the sense defined in §2.1) to a tournament T_7 with group of order 21.

For $n = 9$, we have

$$\max \left\{ g(d)g(9-d) \right\} = 27 .$$

It remains to discuss the case where T_9 is symmetric. Let us consider the tournament T_9 obtained as follows. Replace each node x_1, x_2, x_3 of a 3-cycle by a 3-cycle so that if $x_i \to x_j$ then each node of the 3-cycle replacing x_i dominates each node of the 3-cycle replacing x_j (see Fig. 13). It is easy to see that the tournament T_9 thus defined is symmetric.

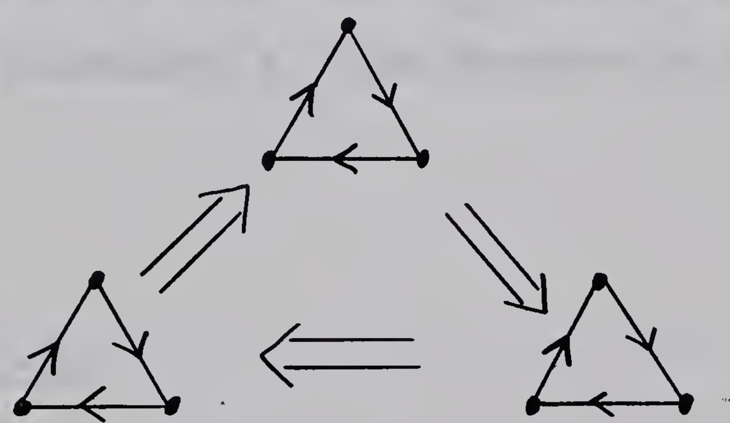


Fig. 13

Since the subgroup of $G(T_9)$ acting on each 3-cycle A,B and C is of order 3 and since A,B and C may be permuted as a 3-cycle it follows that

$$g(T_9) = 3 \cdot 3 \cdot 3 \cdot 3 = 81$$

and therefore,

$$g(9) \geq 81 .$$

It follows from (2.1.5) that

$$g(9) \leq 9(\ g(4)\ )^2 = 81 ,$$

hence

$$g(9) = 81 .$$

Inequality (2.1.5) is therefore best possible in the sense that equality holds for the tournament described above when $n = 9$.

For $n = 10$, a tournament T_{10} with group of order 81 is obtained from the tournament T_9 just described by adding one more independent node.

§2.4 The Case $n = 11$

It can easily be checked, using the fact that $g(9) = 81$, that

$$\max \left\{ g(d)g(11-d) \right\} = 81 .$$

We now show that if T_{11} is symmetric, then $g(T_{11}) \leq 55$ (we will use this fact later). It will follow that $g(11) = 81$.

With A and B defined as in §2.2, let us assume that there exists a 5-orbit $A(5)$ in A . Suppose some set of at least three nodes in B forms a cycle C . It is clear that C cannot dominate $A(5)$ since this would imply that each node of C has score at least 7,

contradicting the symmetry of T_{11}. Similarly, $A(5)$ cannot dominate C because then every node of C would lose to at least 6 nodes (each node of a cycle C must dominate, and be dominated by, at least one other node of C). It follows that there cannot exist any orbit in B which is independent of $A(5)$. A similar argument yields the same conclusion for the case where it is assumed that there exists a 5-orbit in B. Therefore, if there exists a 5-orbit in A (or B) then the nodes of any other orbit in B (or A) must permute in tandem with those of the 5-orbit. It follows that $g(T_{11}) \leq 11 \cdot 5 = 55$ in this case. A group of order 55 is realized by the symmetric tournament T_{11} in which the node x_i dominates the node x_j if and only if $j - i$ is a quadratic residue modulo 11.

We next suppose that there exist independent 3-orbits $A(3)$ and $B(3)$ in A and B and treat first the case where $B(3) \rightarrow A(3)$. Let r and s denote the remaining two nodes in B, and x and y the remaining nodes in A. We may assume without loss of generality that $r \rightarrow s$ and $x \rightarrow y$. Since the nodes of $B(3)$ already have score 5, they must lose to r, s, x, and y. It follows that r, s, x, and y lose to every node of $A(3)$. Since r now has score 5, r loses to x and y. It follows that y beats s and s beats x. This tournament is illustrated in Fig. 14.

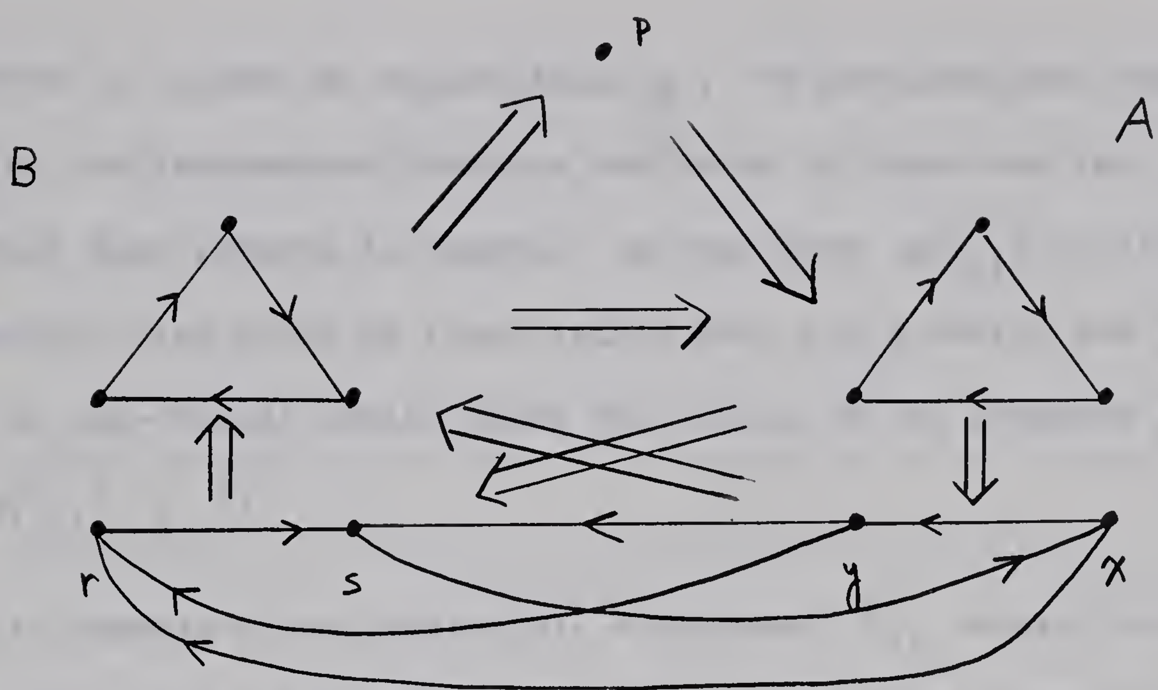


Fig. 14

All nodes certainly have the same score, but p cannot be mapped into r , since the respective tournaments determined by the nodes beaten by p and by r are not isomorphic, contradicting the symmetry of T_{11} .

The other alternative is that $A(3) \to B(3)$. The nodes in $A(3)$ now have score 4 and hence can beat only one more node q . It is clear that $q \to B(3)$ because otherwise q would lose to 6 nodes. The possibilities for q are as follows:

a) $q \in A$ and dominates the remaining point $r \in A$ and some point $s \in B$ $(s \notin B(3))$.

b) $q \in A$ and dominates the remaining points s and $t \in B$.

c) $q \in B$ and dominates the remaining point $s \in B$.

d) $q \in B$ and dominates some point $r \in A$.

In each case the orientation of the remaining edges is determined by the fact that each node must have score five. In all these cases it is readily

verified that p cannot be mapped into q. We conclude that there cannot exist two independent 3-orbits and hence if there are two 3-orbits they must permute in tandem; in this case $g(T_{11}) \leq 11 \cdot 3 = 33$. This inequality also holds if there exists only one 3-orbit, and if there are no non-trivial orbits under the action of the subgroup fixing p then $g(T_{11}) \leq 11$.

An example of an irreducible tournament T_{11} whose group has order 81 consists of a tournament T_9 whose group is of order 81 and two other nodes x and y such that x dominates every node of T_9, y loses to every node of T_9, and y dominates x.

We observe that 11 is the third consecutive number n for which $g(n) = 81$. It is easy to see that for any n, we have

$$g(n+3) \geq 3g(n) .$$

For, we can always construct a tournament T_{n+3} by adding an independent 3-orbit T_3 to a tournament T_n whose group is of order $g(n)$, whence,

$$g(T_{n+3}) = 3g(T_n) .$$

§2.5 <u>Tournaments T_n with Maximum Group for $12 \leq n \leq 22$</u>.

Since

$$\max \left\{ g(d)g(12-d) \right\} = g(3)g(9) = 3 \cdot 81 = 243 ,$$

we see that our tournament T_{12} with group of order 243 may be constructed by adding an independent 3-cycle to the tournament T_9 with group of order 81. For $n = 13$, the required T_{13} is obtained by adding an independent node to the tournament T_{12} just described.

Similar calculations for $n = 14$ show that the tournament T_{14} with maximum group consists of two independent tournaments T_7 both of which have group of order 21.

A tournament T_{15} with maximum possible group is obtained by replacing each node of a 5-cycle by a 3-cycle so that if $p \to q$ in the 5-cycle then each node of the 3-cycle replacing p dominates each node of the 3-cycle replacing q (we say for convenience that such tournaments are obtained "by substitution"; see §2.3 for $n = 9$). The group of this tournament is the Gruppenkranz $C_5[C_3]$ of order $5 \cdot 3^5 = 1215$.

For $n = 16$,

$$\max \left\{ g(d)g(16-d) \right\} = g(7)g(9)$$

$$= 21 \cdot 81 = 1701 .$$

Hence the required T_{16} may be constructed from two independent tournaments T_7 and T_9, both with maximum group. For $n = 17$, we need only add one independent node to the tournament T_{16} just described.

For $n = 18$, (2.1.3) shows that T_{18} with maximum group consists of two independent tournaments T_9 both with group of order 81. Adding one independent node to this tournament T_{18} yields a tournament T_{19} with group of order 6,561. Adding two independent nodes yields the required tournament T_{20}.

A tournament T_{21} with group of order 45,927 is obtained by substitution; namely, each node of a tournament T_7 with maximum group is replaced by a 3-cycle in the manner described for $n = 9$ and $n = 15$. Each 3-cycle may be permuted independently, and the seven 3-cycles may be permuted according to $G(T_7)$. Hence the order of the group is $21 \cdot 3^7 = 7 \cdot 3^8 = 45{,}927$. Adding one independent node to this tournament gives the required tournament T_{22}.

§2.6 The Case n = 23

We prove in this section that $g(23) = 45{,}927$. Using the fact that $g(21) = g(22) = 45{,}927$ it is easily seen that

$$\max \left\{ g(d)g(23 - d) \right\} = 45{,}927 .$$

We now show that if T_{23} is symmetric, then

$$g(T_{23}) < 45{,}927 .$$

With A and B defined as before, let us first suppose that there exists an 11-orbit $A(11)$ in A, and that some set of k nodes

$(3 \leq k \leq 11)$ in B forms a cycle $C(k)$. It is clear that $C(k)$ cannot dominate $A(11)$ since the nodes of $C(k)$ would then have score at least 13. Similarly, $A(11)$ cannot dominate $C(k)$ because this would imply that each node in $C(k)$ loses to at least 12 other nodes. We may therefore conclude that an 11-orbit in A cannot be independent of any cycle, hence any orbit, in B.

To complete the discussion of the case of an 11-orbit, we make use of the following observation.

Lemma 2.6.1

Let a tournament T_{a+b} be composed of two disjoint orbits T_a and T_b. If T_a and T_b are not independent, then $(a,b) \neq 1$.

Proof:

Suppose T_a and T_b are not independent. Since every node of T_a is similar to every other node of T_a (and likewise for T_b), let each node of T_a dominate β nodes of T_b and let each node of T_b dominate α nodes of T_a. Then we have $0 < \alpha < a$ and $0 < \beta < b$. Since there are ab edges joining nodes of T_a and T_b, it follows that

$$a\beta = b(a - \alpha) = M \quad ,$$

a common multiple of a and b. If $(a,b) = 1$ it must be that $a\beta = M \geq ab$ whence $\beta \geq b$, a contradiction. Hence $(a,b) \neq 1$.

It is an immediate consequence of Lemma 2.6.1 that the only possible non-trivial non-independent orbit that could exist in B if A(11) exists in A is an 11-orbit B(11) . It follows from Lemma 2.1.1 that in this case

$$g(T_{23}) \leq \frac{23 \cdot 55 \cdot 55}{5} < 45{,}927 .$$

A similar result holds in the case where it is assumed that there exists an 11-orbit in B .

Let us next suppose that there exists a 9-orbit A(9) in A . Firstly, let us assume there exists a 9-orbit B(9) in B . It is clear that B(9) cannot dominate A(9) since then every node of B(9) would have score at least 14. Similarly, A(9) cannot dominate B(9) since then every node of A(9) would have score at least 13. Hence B(9) cannot be independent of A(9). Therefore, if B(9) permutes in tandem with A(9) we have

$$g(T_{23}) \leq 23 \cdot 81 < 45{,}927 .$$

(Although B(9) is not independent of A(9), B(9) need not necessarily move in tandem with A(9), but this case will be dealt with later.) Secondly, suppose there exists a 7-orbit B(7) in B . Clearly B(7) cannot dominate A(9) since then every node of B(7) would have score at least 13. Similarly, A(9) cannot dominate B(7) since then every node of B(7) would lose to 12 nodes. Hence B(7) is not independent of A(9). For this case we therefore have the inequality

$$g(T_{23}) \leq 23 \cdot 81 \cdot 3 < 45,927 .$$

Thirdly, let us assume that there exists a 5-orbit $B(5)$ in B. Now $B(5)$ cannot dominate $A(9)$ because then every node of $B(5)$ would have score at least 12. If $A(9) \rightarrow B(5)$ then every node of $A(9)$ now has score 9 ; furthermore every node of $B(5)$ now has lost to 11 nodes, hence dominates every other node. It is clear that $A(9)$ cannot dominate either remaining node in A, since this node would then lose to at least 15 nodes. Therefore $A(9)$ dominates two nodes of the set S of six nodes in B which are not in $B(5)$. It follows that S could contribute at most three independent automorphisms and in this case we have the inequality

$$g(T_{23}) \leq 23 \cdot 81 \cdot 5 \cdot 3 < 45,927 .$$

Fourthly, we consider the case where there are independent 3-orbits in B. Let us therefore assume that there exist three 3-orbits $B_1(3)$, $B_2(3)$, $B_3(3)$ independent of each other in B and each independent of $A(9)$. We may suppose without loss of generality that $B_1(3) \rightarrow B_2(3) \rightarrow B_3(3)$. It is clear that $B_2(3)$ cannot dominate $A(9)$ since each node of $B_2(3)$ would then have score at least 14. But $A(9)$ cannot dominate $B_2(3)$ either, because each node of $B_2(3)$ would then lose to at least 13 nodes. We may conclude that there cannot exist three independent 3-orbits in B. If there are two independent 3-orbits in B we have

$$g(T_{23}) \leq 23 \cdot 81 \cdot 9 < 45,927 .$$

Clearly this inequality also holds if there is only one 3-orbit in B, and hence the discussion of the case of an independent 9-orbit in A is completed. Similar considerations take care of the case where an independent 9-orbit is assumed to exist in B.

It is possible for there to exist non-independent 9-orbits in A and B such that their nodes need not necessarily move in tandem; certain sub-orbits of both orbits may permute independently, whence an additional contribution to $g(T_{23})$ will arise. We say that under such conditions, the nodes permute "co-operatively". We shall presently show, however, that in such cases the symmetry of T_{23} is violated.

The (unique) 9-orbit with group of order 81 was seen to be the tournament obtained by replacing each node x_1, x_2, x_3 of a 3-cycle by a 3-cycle $A_1(3)$, $A_2(3)$, $A_3(3)$ such that if $x_i \to x_j$ then $A_i(3) \to A_j(3)$ for $i,j = 1,2,3$. We label the 3-cycles of the 9-orbits in A and B respectively by $A_1(3)$, $A_2(3)$, $A_3(3)$ and $B_1(3)$, $B_2(3)$, $B_3(3)$. It is not difficult to see that the only possibilities we need to consider are as follows:

(1) $A_i(3) \to B_i(3)$ for $i = 1,2,3$ and $B_i(3) \to A_j(3)$ if $i \neq j$.

(2) $B_i(3) \to A_i(3)$ for $i = 1,2,3$ and $A_i(3) \to B_j(3)$ if $i \neq j$.

Let us first consider case (1). We label the two remaining points in A by x and y and suppose that $x \to y$. Each node of a 3-cycle $B_i(3)$ now dominates 11 nodes (namely p, one other node of $B_i(3)$,

each node of one of the two remaining 3-cycles $B_j(3)$, and each node of two of the 3-cycles in A) and hence must lose to x , y and the two remaining nodes in B. Each node of each 3-cycle $A_i(3)$ has score 7 now and hence must also beat x , y and the remaining two nodes of B . It follows that x beats one of the two remaining nodes of B, and y beats both of them. This yields the scheme shown in Fig. 15.

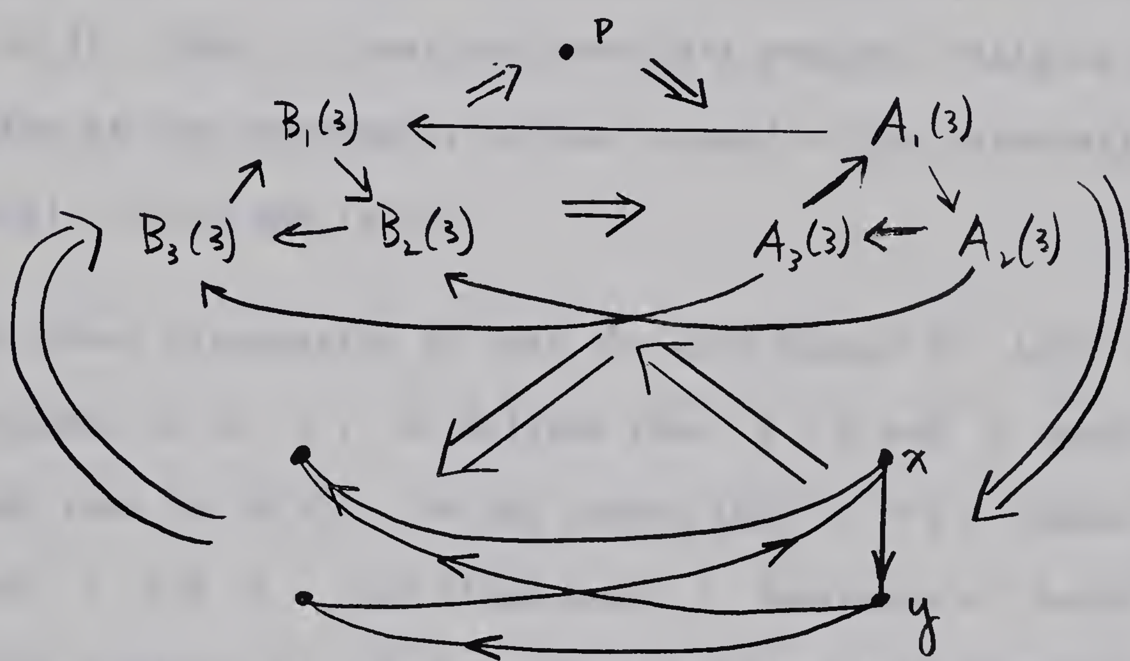


Fig. 15

Since the fixed node p beats a 9-cycle that beats two other nodes, it follows from symmetry that every node must have this property. It is easily seen, however, that y does not have this property.

We next consider case (2). Each node of each 3-cycle $A_i(3)$ now has score 10 and hence can beat only one more node. We may assume that this node dominates $B_1(3)$, $B_2(3)$ and $B_3(3)$. We label the remaining nodes in A by x and y and the two remaining nodes in B by r and s .

If the singular node beaten by $A(9)$ is in A, we may assume without loss of generality that x is this node. It follows from the definition of x that r,s and y each dominate the 9-orbit $A(9)$ and each lose to the 9-orbit $B(9)$. We may assume without loss of generality that $r \to s$. Hence, r loses to both x and y. Our fixed node p dominates a 9-orbit, one node which dominates the 9-orbit, and one node which loses to it. That r does not have this property violates the assumed symmetry of the tournament, without regard to the orientation of the edges (x,y), (x,s) and (s,y).

The other alternative is that the node beaten by $A(9)$ is in B; we may suppose it is r. It follows that s, x and y each dominate $A(9)$ and each lose to $B(9)$. We may assume that $x \to y$. Hence x dominates both r and s. Our fixed node p dominates a 9-orbit and two nodes which dominate it. That y does not have this property violates the assumed symmetry of the tournament, without regard to the orientation of the edges (r,s), (r,x) and (s,x). We conclude that if the tournament is symmetric, we cannot have two non-independent 9-orbits whose nodes permute co-operatively. Appealing to Lemma 2.1.1, we therefore have

$$g(T_{23}) \leq \frac{23 \cdot 27 \cdot 27 \cdot 3}{3} < 45{,}927 .$$

We make the obvious remark that if every orbit in B is trivial or moves in tandem with $A(9)$ then for such a tournament we would have

$$g(T_{23}) \leq 23 \cdot 81 < 45{,}927 .$$

Let us now examine the cases where there exists a 7-orbit $A(7)$ in A . Again, if every other orbit in T_{23} is trivial or moves in tandem with $A(7)$ then

$$g(T_{23}) \leq 23 \cdot 21 < 45,927 \quad .$$

We first assume, therefore, that there exists an independent 7-orbit $B(7)$ in B , and independent 3-orbits $A(3)$ and $B(3)$ in A and B . We label the remaining node in A and in B by a and b . Suppose that $B(7) \rightarrow A(7)$. Since each node in $A(7)$ now loses to 11 nodes (namely each node of $B(7)$, the fixed node p , and 3 nodes of $A(7)$), $A(7)$ must dominate all other nodes of the tournament. Each node of $B(7)$ has score 11, hence $B(7)$ loses to all remaining nodes of the tournament. Since each node of $A(3)$ has now lost to 9 nodes it follows that $A(3)$ loses to a and to b . Each node of $B(3)$ has score 9 , hence $B(3)$ must dominate a and b . It follows that $a \rightarrow b$. This yields the scheme shown in Fig. 16.

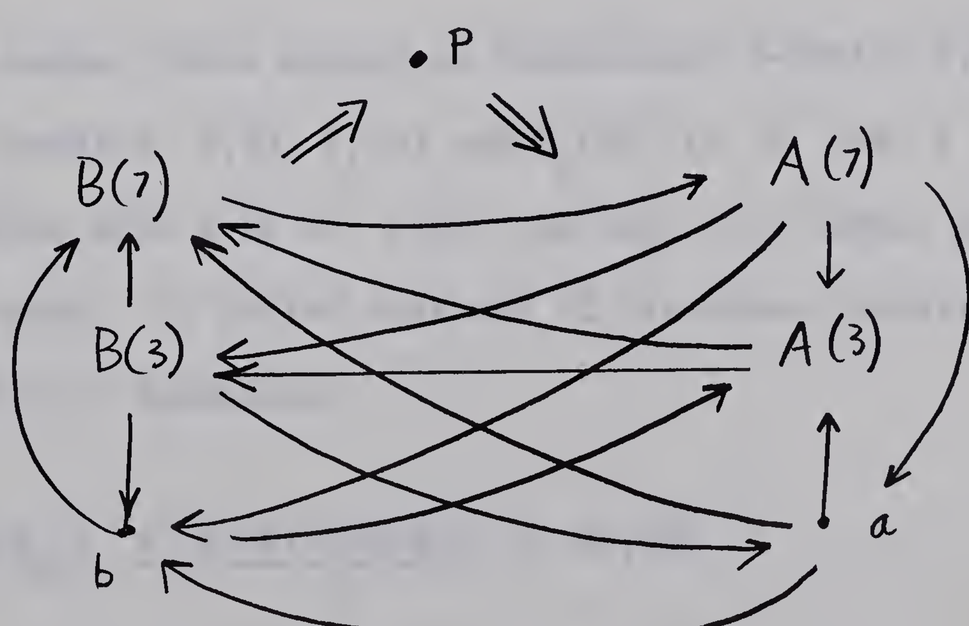


Fig. 16

Therefore our fixed node p, hence by symmetry every node, dominates a 7-orbit and a 3-orbit that loses to the 7-orbit. Since node a does not have this property the symmetry of T_{23} is violated and it follows that in this case

$$g(T_{23}) \leq \frac{23 \cdot 21 \cdot 21 \cdot 3 \cdot 3}{3} < 45,927 .$$

Suppose now that $A(7) \to B(7)$. It follows immediately that both $A(3)$ and $B(3)$ dominate $A(7)$ and lose to $B(7)$. Since each node of $A(3)$ now has score 8, it must be that $A(3) \to B(3)$. Each node of $B(3)$ has score 9, hence $B(3)$ dominates both a and b. Node a or b both dominates $B(7)$. They both dominate $A(3)$, and also $a \to b$. Our fixed node p, hence every node, dominates a 7-orbit and a 3-orbit that dominates the 7-orbit. That node a (or b) hasn't this property violates the symmetry of T_{23}. The inequality in this case is the same as that stated above.

We now assume there exists an independent 5-orbit $B(5)$ in B, and independent 3-orbits $A(3)$, $B_1(3)$ and $B_2(3)$ in A and B. Suppose $A(7) \to B(5)$. Since each node of $B(5)$ has lost to 9 nodes, they can lose to only two more. It follws that one of the three 3-orbits is not independent of $B(5)$. Therefore,

$$g(T_{23}) \leq \frac{23 \cdot 21 \cdot 3 \cdot 5 \cdot 3 \cdot 3}{3} < 45,927 .$$

Suppose that $B(5) \to A(7)$. Each node of $A(7)$ can lose to only two more nodes hence we reach the same conclusion and inequality as before.

If there are two independent 5-orbits $B_1(5)$ and $B_2(5)$ in B we have

$$g(T_{23}) \leq 23\cdot 21\cdot 5\cdot 5\cdot 3 < 45{,}927 ,$$

and it is clear that if the longest orbits in B are of length three then

$$g(T_{23}) \leq 23\cdot 21\cdot 3\cdot 3\cdot 3\cdot 3 < 45{,}927 .$$

Similar arguments to those used above will take care of the dual case where the 7-orbit is in B .

We may now assume that the only orbits in A or B are of length 3 or 5 . We first consider the following case: there exist orbits $A(5)$, $A_1(3)$, $A_2(3)$ in A and $B(5)$, $B_1(3)$, $B_2(3)$ in B . Each node of $A_1(3)$ must lose to 9 more nodes. Hence $B_1(3)$, $B_2(3)$ and $A_2(3)$ each dominate $A_1(3)$. But each node of $A_2(3)$ must lose to 9 more nodes and clearly this cannot be done without destroying the independence of one of the 5-orbits. It follows that

$$g(T_{23}) \leq \frac{23\cdot 5\cdot 3\cdot 3\cdot 5\cdot 3\cdot 3}{3} < 45{,}927 .$$

We next consider the case where the subgroup acting on A has order at most $5 \cdot 3 \cdot 3 = 45$ and the subgroup acting on B has order at most $3 \cdot 3 \cdot 3 = 27$. Then

$$g(T_{23}) \leq 23 \cdot 45 \cdot 27 < 45{,}927 .$$

We have now exhausted the cases for a symmetric tournament T_{23} and therefore the proof that $g(23) = 45{,}927$ is complete.

An example of an irreducible tournament T_{23} whose group has order $45{,}927$ is one consisting of a tournament T_{21} whose group is of order $45{,}927$ and two other nodes x and y such that x dominates every node of T_{21}, y loses to every node of T_{21}, and y dominates x.

§2.7 Tournaments T_n with Maximum Group for $24 \leq n \leq 26$

Since

$$\begin{aligned} \max \left\{ g(d)g(24 - d) \right\} &= g(3)g(21) \\ &= 3 \cdot 45{,}927 = 137{,}781 \end{aligned}$$

we see that the tournament T_{24} with maximum group is obtained by adding an independent 3-cycle to the tournament T_{21} described in §2.5. Adding an independent node to the tournament T_{24} thus constructed yields a tournament T_{25} with group of required order.

Similar calculations for $n = 26$ show that a tournament T_{26} with group of order 229,635 consists of the tournament T_{21} mentioned above to which an independent tournament T_5 with group of order 5 has been adjoined.

§2.8 <u>The Case $n = 27$</u>

We now show that $g(27) = 3^{13} = 1{,}594{,}323$.

It is readily verified that

$$\max \left\{ g(d)g(27 - d) \right\} = 531{,}441 .$$

It remains to consider the case where T_{27} is symmetric. In this case we have, using (2.1.5), that

$$g(27) \leq 27 [g(13)]^2 = 1{,}594{,}323 .$$

Suppose we are given three tournaments A , B and C each of which is a symmetric tournament on 9 nodes whose group is of order 81, and suppose that $A \to B$, $B \to C$ and $C \to A$. (This is equivalent to replacing each node of a tournament T_9 with group of order 81 by a 3-cycle such that if $p \to q$ in T_9 then each node of the 3-cycle replacing p dominates each node of the 3-cycle replacing q.) The tournament T_{27} thus defined is easily seen to be symmetric. Since each of the subgroups of $G(T_{27})$ acting on A, B, and C respectively have order 81, and since the tournaments A, B and C can be permuted as a 3-cycle it follows that

$$g(T_{27}) = 81 \cdot 81 \cdot 81 \cdot 3 = 3^{13} = 1{,}594{,}323 \ .$$

Therefore,

$$g(27) \geq 1{,}594{,}323 \ .$$

This suffices to prove that $g(27) = 1{,}594{,}323$.

We have seen that inequality (2.1.5) is best possible in the sense that equality holds for tournaments of the type described for $n = 3, 9, 27$. We conjecture that equality holds for such tournaments for all higher powers of three as well.

CHAPTER III

BOUNDS FOR THE MAXIUMUM ORDER OF THE GROUP OF A TOURNAMENT

§3.1 Introduction

We now use the results of Chapter II to obtain bounds for $g(n)$ by induction. These bounds will enable us to establish the existence of $\lim g(n)^{1/n}$.

We remark here that recently Weinberg [50] has shown that if G is a 3-connected planar graph with e edges, then $|\Gamma(G)| \leq 4e$. This result has been generalized by Sabidussi [48].

§3.2 An Upper Bound

In this section we prove by induction that

$$g(n) \leq \frac{.45(2.03)^n}{n} \quad , \quad \text{for} \quad n \geq 13 \; . \tag{3.2.1}$$

It is not difficult to verify that this inequality holds when $13 \leq n \leq 26$ by using the exact values of $g(n)$ given in Table 1 .

Consider an arbitrary tournament T_n , where $n \geq 27$, and recall the definitions of D , d , T_d , and T_{n-d} given in §2.1. If $13 \leq d \leq n - 13$, then it follows from (2.1.1) and the induction hypothesis that

$$g(T_n) \leq \frac{.45(2.03)^d}{d} \cdot \frac{.45(2.03)^{n-d}}{n-d}$$

$$\leq \frac{.45(2.03)^n}{n} \left(\frac{.45n}{13(n-13)}\right) < \frac{.45(2.03)^n}{n} .$$

If $d = 1,3,5,7,9,11$ or $n-1, n-3, n-5, n-7, n-9, n-11$ then

$$g(T_n) \leq g(d) \cdot \frac{.45(2.03)^{n-d}}{n-d}$$

$$= \frac{.45(2.03)^n}{n} \left(\frac{g(d)n}{(2.03)^d(n-d)}\right) .$$

Simple computations show that for all these values of d (and $n \geq 27$), we have

$$\frac{g(d)n}{(2.03)^d(n-d)} \leq 1 ;$$

hence,

$$g(T_n) \leq \frac{.45(2.03)^n}{n} .$$

The same argument holds when $n-d = 2,4,6,8,10,12$.

If $d = n$, then from (2.1.5) we have

$$g(T_n) \leq n\left(\frac{.45(2.03)^{\frac{n-1}{2}}}{\frac{n-1}{2}}\right)^2$$

$$= \frac{.45(2.03)^n}{n}\left(\frac{4(.45)}{2.03}\left(\frac{n}{n-1}\right)^2\right) < \frac{.45(2.03)^n}{n} .$$

Notice that if $n \geq 27$, then $\frac{1}{2}(n-1) \geq 13$, so we are certainly entitled to apply the induction hypothesis to $g\left(\frac{1}{2}(n-1)\right)$. This suffices to complete the proof of inequality (3.2.1) by induction.

Stronger forms of inequality (3.2.1) can presumably be obtained by the same type of argument if one is willing to treat more cases separately. In view of the lengthy, yet incomplete, discussion in Chapter II, however, the improvement would hardly seem worth the effort.

We conjecture (see Table 1) that

$$g(n) \leq \sqrt{3}^{\,n-1}$$

with equality holding if and only if $n = 3^k$, $k = 0,1,2,\ldots$.

§3.3 Proof of the Existence of the Limit

Our object in this section is to prove the following result.

Theorem 3.3.1

The limit of $g(n)^{1/n}$ as n tends to infinity exists and lies between $\sqrt{3}$ and 2.03 , inclusive.

If T_a and T_b are two arbitrary tournaments, consider the tournament T_{ab} obtained by replacing each node of T_a by a copy of T_b ; if the node p dominates the node q in T_a originally, then in T_{ab} each node of the tournament that replaces p dominates each node of the tournament that replaces q . (Such "substitution tournaments" have been mentioned before ; for examples see §2.3, §2.5 and §2.7.) It is not difficult to see that the orders of the groups of T_a , T_b and T_{ab} satisfy the inequality

$$g(T_{ab}) \geq g(T_a)\,[g(T_b)]^a \quad .$$

Therefore,

$$g(ab) \geq g(a)\,[g(b)]^a \; , \tag{3.3.1}$$

for all integers a and b . In particular, since $g(3) = 3$, it follows by induction that

$$g(n) \geq \sqrt{3}^{\,n-1} \quad \text{if} \quad n = 3^k \; , \; k = 0,1,2,\ldots \; . \tag{3.3.2}$$

Hence,

$$\limsup\; g(n)^{1/n} \geq \sqrt{3} \quad . \tag{3.3.3}$$

It is an immediate consequence of inequality (3.2.1) that

$$\limsup\; g(n)^{1/n} \leq 2.03 \; . \tag{3.3.4}$$

We now use inequality (3.3.1) to prove the following result.

Lemma 3.3.1

If $g(m)^{1/m} > \gamma$, then $g(n)^{1/n} > \gamma - \epsilon$ for any positive ϵ and all sufficiently large n.

Proof:

We assume $\gamma > 1$ since the result is obvious otherwise. Let ℓ be the least positive integer such that $\gamma^{-1/\ell} > 1 - \epsilon/\gamma$. Every sufficiently large integer n can be written in the form $n = km + t$, where $k > \ell$ and $0 \le t < m$. Then

$$\begin{aligned} g(n)^{1/n} &= g(km+t)^{\frac{1}{km+t}} \ge g(km)^{\frac{1}{m(k+1)}} \\ &\ge [g(m)^{1/m}]^{\frac{k}{k+1}} > \gamma^{\frac{k}{k+1}} \\ &> \gamma^{\frac{\ell}{\ell+1}} > \gamma^{1-1/\ell} \\ &\ge \gamma(1 - \frac{\epsilon}{\gamma}) = \gamma - \epsilon \end{aligned}$$

as required.

Let $\beta = \lim\sup g(n)^{1/n}$. We know from (3.3.3) and (3.3.4) that $\sqrt{3} \le \beta \le 2.03$. For every positive ϵ there exists an integer m such that

$$g(m)^{1/m} > \beta - \epsilon \quad ,$$

by definition of β . But then, according to Lemma 3.3.1,

$$g(n)^{1/n} > \beta - 2\epsilon$$

for all sufficiently large n . Hence,

$$\liminf g(n)^{1/n} > \beta - 2\epsilon$$

for every positive ϵ . It follows that

$$\liminf g(n)^{1/n} = \limsup g(n)^{1/n} . \tag{3.3.5}$$

Theorem 3.3.1 now follows from statements (3.3.3), (3.3.4) and (3.3.5).

§3.4 Concluding Remarks

The main purpose of this section is to show that the problem of determining $g(n)$ is equivalent to a strictly group-theoretical problem.

Let G be a subgroup of S_n , every element of which may be expressed as a product of disjoint cycles of odd length; in particular, no element of G interchanges two letters. Let G act on the vertices of the ordinary complete n-graph K_n . Then G induces an equivalence relation on the edges of K_n ; two edges (a,b) and (x,y) are said to be <u>G-equivalent</u> in K_n if and only if there exists a permutation $g \in G$ mapping the set $\{a,b\}$ onto the set $\{x,y\}$. Assign an arbitrary orientation to one edge from each equivalence class, and orient the images of these edges under G in the same way. It follows that the

orientation of all edges in K_n are uniquely determined. This procedure defines a tournament T_n , and it is easy to see that G is a subgroup of $G(T_n)$.

The following lemma characterizes the groups G .

<u>Lemma 3.4.1</u>

Let G be a subgroup of S_n . Then every element of G may be expressed as the product of disjoint cycles of odd length if and only if the order of G is odd.

<u>Proof</u>: If every $g \in G$ may be expressed as the product of disjoint cycles of odd length, then the order of g , being the least common multiple of the lengths of the cycles of g (see Wielandt [51] p.3), is odd. It follows that the order of G is odd, since if $|G| = 2k$ for some integer k , then G has an element of order 2.

Conversely, let the order of G be odd. Suppose some $g \in G$ may be written as follows:

$$g = g_1 g_2$$

where

$$g_1 = \text{(product of cycles of even length)}$$

$$g_2 = \text{(product of cycles of odd lengths } \ell_1, \ell_2, \ldots, \ell_r\text{)}.$$

Let $[\ell_1, \ell_2, \ldots, \ell_r]$ denote the least common multiple of the numbers $\ell_1, \ell_2, \ldots, \ell_r$.

Then

$$g^{[\ell_1,\ell_2,\ldots,\ell_r]} = g_1^{[\ell_1,\ell_2,\ldots,\ell_r]} g_2^{[\ell_1,\ell_2,\ldots,\ell_r]}$$

$$= g_1^{[\ell_1,\ell_2,\ldots,\ell_r]} = g_1' \ .$$

It is clear that g_1' is another product of cycles of even length, hence the order of g_1' is even. By Lagrange's theorem, this contradicts the definition of G . This suffices to complete the proof of Lemma 3.4.1.

Since $g(T_n)$ is always odd, if G is chosen to be as large as possible in S_n , the inclusion relation

(3.4.1) $\quad G \leq G(T_n) < S_n$

implies that the group of the tournament obtained from G by the above procedure is isomorphic to G . It is an immediate consequence of this observation that if G is the largest subgroup of odd order contained in S_n , then $|G| = g(n)$. Thus, the problem of determining $g(n)$ is equivalent to determining the order of the largest subgroup of odd order in S_n .

The above comments imply the following result.

<u>Theorem 3.4.1</u>

If $n \geq 3$, let G be the largest subgroup of odd order in S_n. Then there exists a tournament with n nodes whose automorphism group is

permutationally isomorphic to G .

In determining the order of G , it is sufficient to consider the case where G is transitive. For, if G is not transitive, then G partitions the set D on which it acts into distinct orbits $D_1, D_2, \ldots, D_k$. Each $g \in G$ induces a permutation g^i on D_i $(i = 1,2,\ldots,k)$ defined as follows:

$$(3.4.2) \qquad \begin{aligned} g^i(d) &= g(d) \qquad && d \in D_i \\ g^i(d) &= d && d \notin D_i \end{aligned} \quad .$$

The set $\{g^i \mid g \in G\}$ is called the <u>i^{th} transitive constituent G^i</u> of G on D_i . It is clear that $g \to g^i$ is a homomorphism from G to G^i . Now each $g \in G$ is the product of permutations g^i , so we certainly have

$$(3.4.3) \qquad G \leq \prod_{i=1}^{k} G^i \quad .$$

Each G^i is a permutation group of odd order, so $\prod_{i=1}^{k} G^i$ is also; it follows from the maximality of G that

$$(3.4.4) \qquad G = \prod_{i=1}^{k} G^i$$

It is an easy consequence of (3.4.4) that

$$(3.4.5) \qquad |G| = \prod_{i=1}^{k} |G^i| .$$

The subgroup G^i is the largest transitive subgroup of G acting on D_i . If $|D_i|$ denotes the length of the orbit D_i , let $\bar{g}$ be the function defined as follows:

$$(3.4.6) \qquad \bar{g}(|D_i|) = |G^i| .$$

Thus, $\bar{g}(|D_i|)$ is the order of the largest transitive subgroup acting on D_i . Writing $\pi_i = |D_i|$, the orbit structure $D_1, D_2, \ldots, D_k$ corresponds to the partition $\pi : \pi_1 + \pi_2 + \ldots + \pi_k = n$. Hence (3.4.5) becomes

$$(3.4.7) \qquad |G| = \prod_{i=1}^{k} \bar{g}(\pi_i) ,$$

corresponding to the partition π . Maximizing over all partitions of n (i.e. all possible orbit structures of G) we have

$$(3.4.8) \qquad g(n) = \max_{\pi} \prod_{i=1}^{k(\pi)} \bar{g}(\pi_i) .$$

We have therefore reduced the problem to that of finding $\bar{g}(n)$, the order of the largest transitive subgroup of odd order in S_n .

We remark in closing that, in obtaining bounds for $\bar{g}(n)$, it is sufficient to obtain bounds for the order of the largest non-abelian

transitive subgroup of odd order in S_n , since in a recent paper, Bercov and Moser [1] have determined the order of the largest abelian subgroup of S_n .

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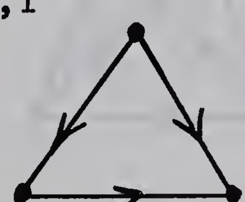
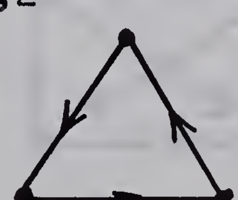
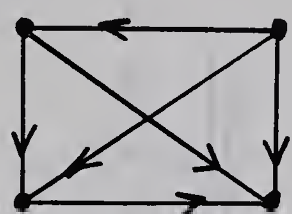
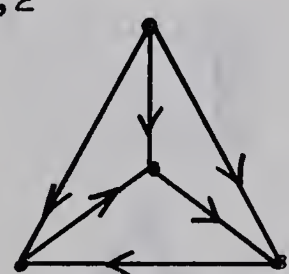
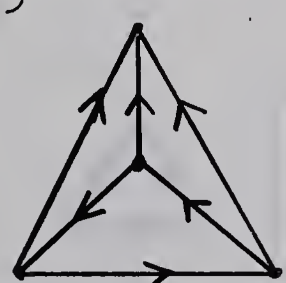
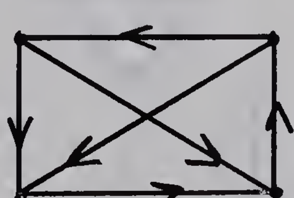
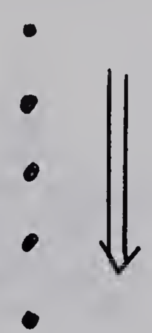
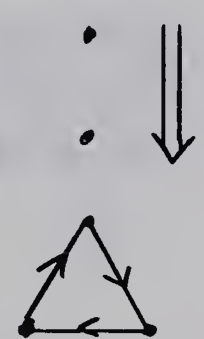
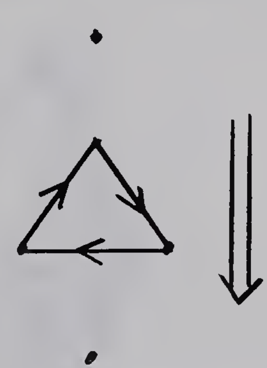
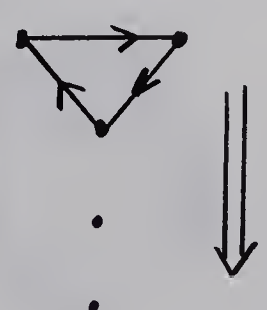
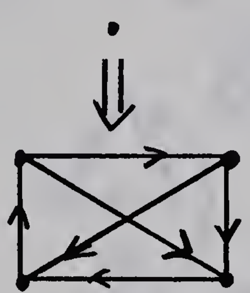
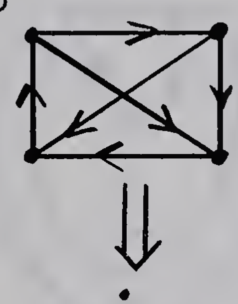
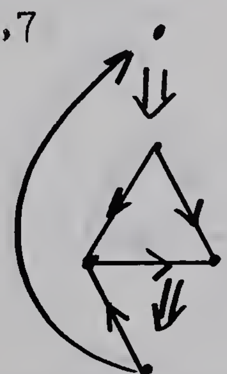
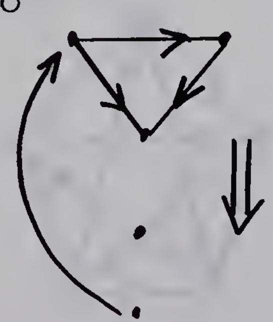
APPENDIX

TOURNAMENTS T_n , $n \leq 6$, AND THEIR GROUPS

Score Vector	Tournament	Number of Labellings	Group	Remarks
$n = 1$	T_1 	1	I	
$n = 2$ (0,1)	T_2 	2	I	
$n = 3$ (0,1,2)	$T_{3,1}$ 	6	I	
(1,1,1)	$T_{3,2}$ 	2	C_3	

Score Vector	Tournament	Number of Labellings	Group	Remarks
$n = 4$ (0,1,2,3)	$T_{4,1}$ 	24	I	
(1,1,1,3)	$T_{4,2}$ 	8	C_3	
(0,2,2,2)	$T_{4,3}$ 	8	C_3	Dual to $T_{4,2}$
(1,1,2,2)	$T_{4,4}$ 	24	I	

Score Vector	Tournament	Number of Labellings	Group	Remarks
n = 5 (0,1,2,3,4)	$T_{5,1}$ 	120	I	
(1,1,1,3,4)	$T_{5,2}$ 	40	C_3	
(0,2,2,2,4)	$T_{5,3}$ 	40	C_3	
(0,1,3,3,3)	$T_{5,4}$ 	40	C_3	Dual to $T_{5,2}$

Score Vector	Tournament	Number of Labellings	Group	Remarks
(1,1,2,2,4)	$T_{5,5}$ 	120	I	
(0,2,2,3,3)	$T_{5,6}$ 	120	I	Dual to $T_{5,5}$
(1,1,2,3,3)	$T_{5,7}$ 	120	I	
(1,1,2,3,3)	$T_{5,8}$ 	120	I	

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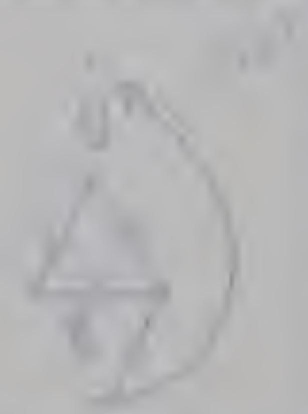
	[illegible]	[illegible]		[illegible]
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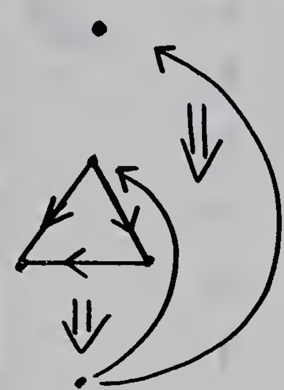
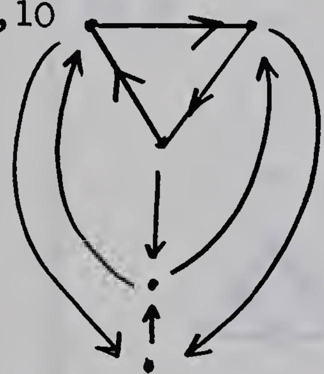
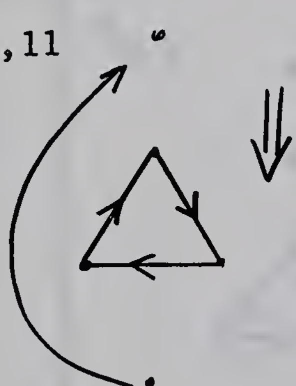
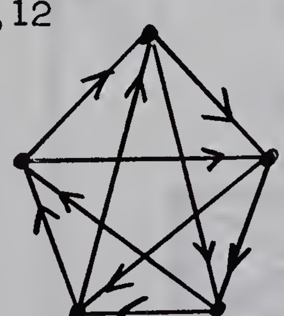
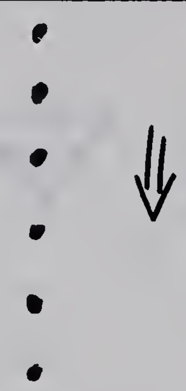
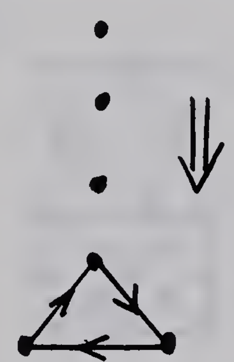
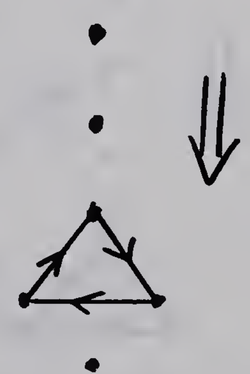
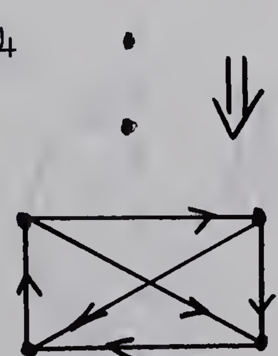
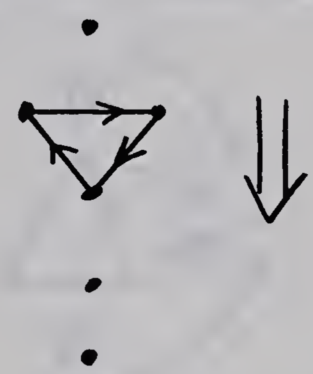
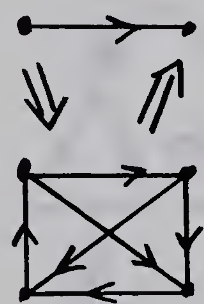
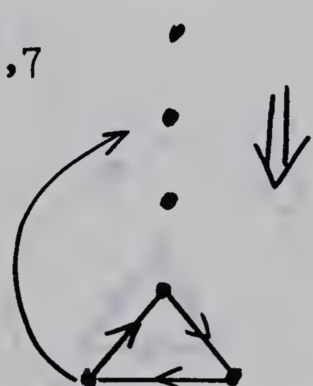
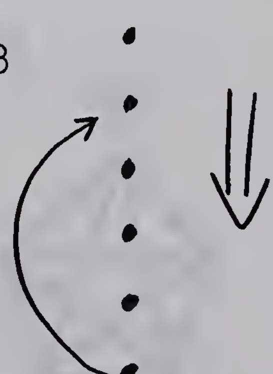
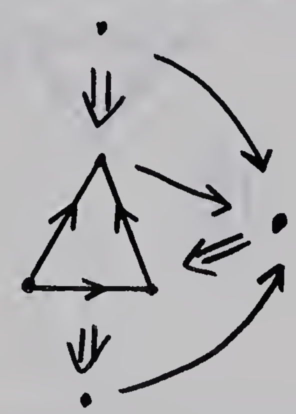
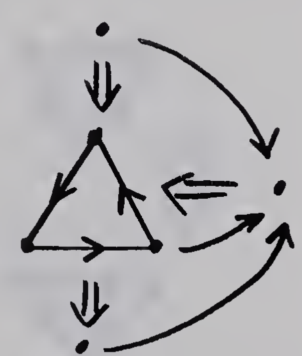
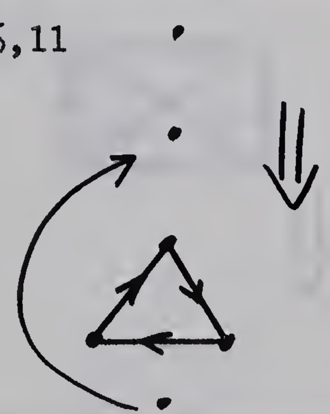
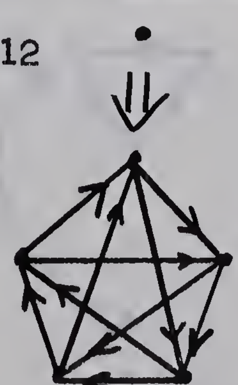
Score Vector	Tournament	Number of Labellings	Group	Remarks
(1,2,2,2,3)	$T_{5,9}$ 	120	I	
(1,2,2,2,3)	$T_{5,10}$ 	120	I	
(1,2,2,2,3)	$T_{5,11}$ 	40	C_3	
(2,2,2,2,2)	$T_{5,12}$ 	24	C_5	-the only symmetric T_5

Score Vector	Tournament	Number of Labellings	Group	Remarks
$n = 6$ (0,1,2,3,4,5)	$T_{6,1}$ 	720	I	
(1,1,1,3,4,5)	$T_{6,2}$ 	240	C_3	
(0,2,2,2,4,5)	$T_{6,3}$ 	240	C_3	
(1,1,2,2,4,5)	$T_{6,4}$ 	720	I	

Score Vector	Tournament	Number of Labellings	Group	Remarks
(0,1,3,3,3,5)	$T_{6,5}$ 	240	C_3	Dual to $T_{6,3}$
(0,2,2,3,3,5)	$T_{6,6}$ 	720	I	
(1,1,2,3,3,5)	$T_{6,7}$ 	720	I	
(1,1,2,3,3,5)	$T_{6,8}$ 	720	I	

Score Vecot	Tournament	Number of Labellings	Group	Remarks
(1,2,2,2,3,5)	$T_{6,9}$ 	720	I	
(1,2,2,2,3,5)	$T_{6,10}$ 	720	I	
(1,2,2,2,3,5)	$T_{6,11}$ 	240	C_3	
(2,2,2,2,2,5)	$T_{6,12}$ 	144	C_5	

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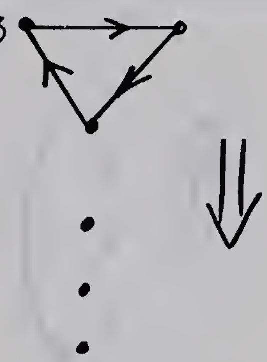
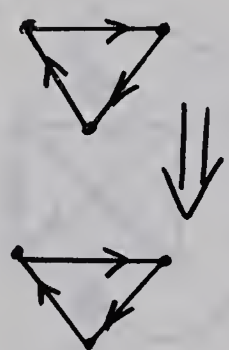
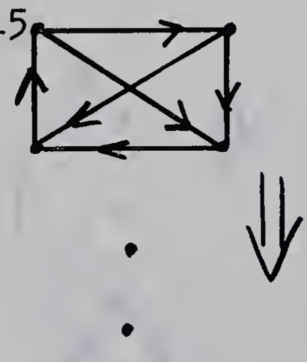
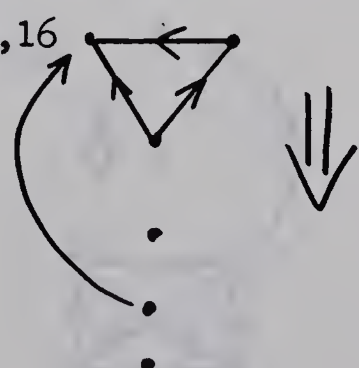
Score Vector	Tournament	Number of Labellings	Group	Remarks
(0,1,2,4,4,4)	$T_{6,13}$ 	240	C_3	Dual to $T_{6,2}$
(1,1,1,4,4,4)	$T_{6,14}$ 	80	$C_3 \times C_3$	Only T_6 with group of order g(6) .
(0,1,3,3,4,4)	$T_{6,15}$ 	720	I	Dual to $T_{6,4}$
(0,2,2,3,4,4)	$T_{6,16}$ 	720	I	Dual to $T_{6,7}$

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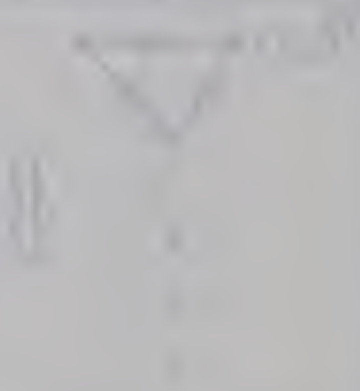
[illegible]	[illegible]	[illegible]		[illegible]
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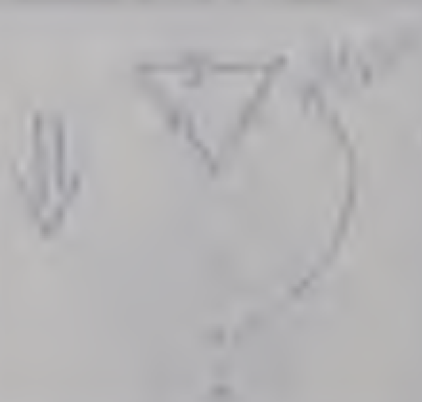
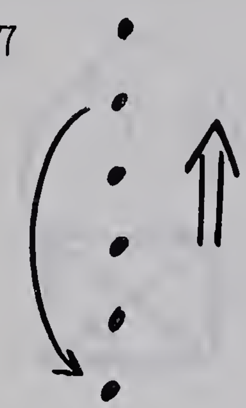
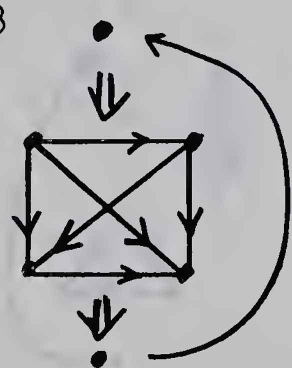
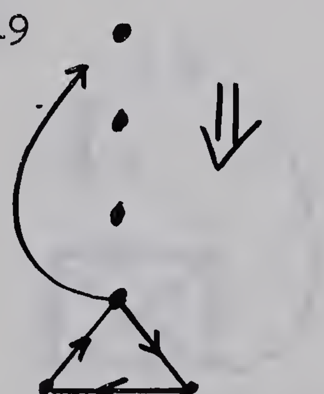
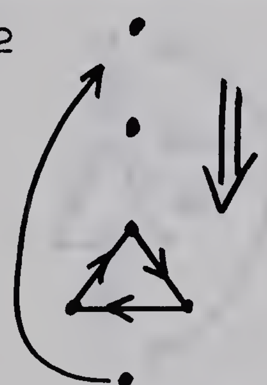
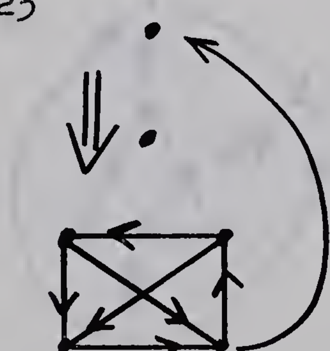
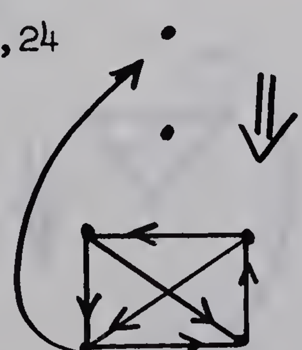
[illegible]	[illegible]	[illegible]		[illegible]

Score Vector	Tournament	Number of Labellings	Group	Remarks
(0,2,2,3,4,4)	$T_{6,17}$ 	720	I	Dual to $T_{6,8}$
(1,1,2,3,4,4)	$T_{6,18}$ 	720	I	
(1,1,2,3,4,4)	$T_{6,19}$ 	720	I	
(1,1,2,3,4,4)	$T_{6,20}$ 	720	I	

Score Vector	Tournament	Number of Labellings	Group	Remarks
(1,1,2,3,4,4)	$T_{6,21}$ 	720	I	
(1,2,2,2,4,4)	$T_{6,22}$ 	240	I	
(1,2,2,2,4,4)	$T_{6,23}$ 	720	I	
(1,2,2,2,4,4)	$T_{6,24}$ 	720	I	

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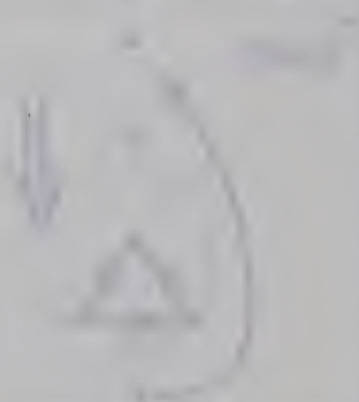
	[illegible]	[illegible]		[illegible]
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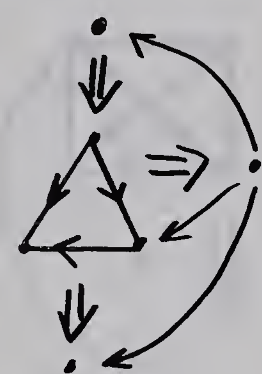
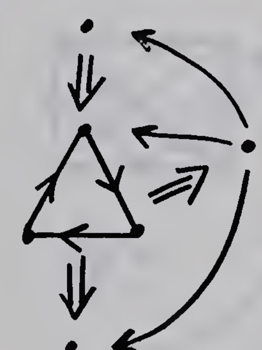
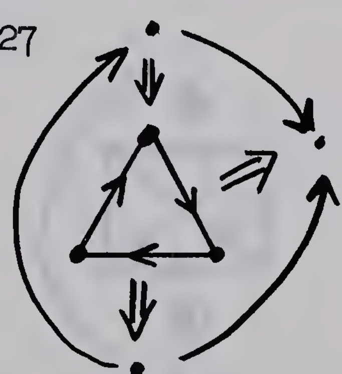
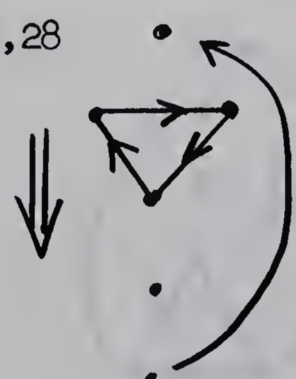
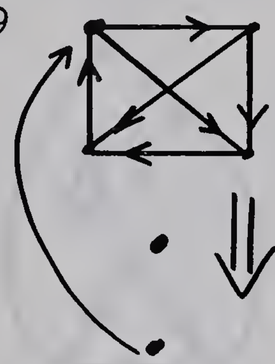
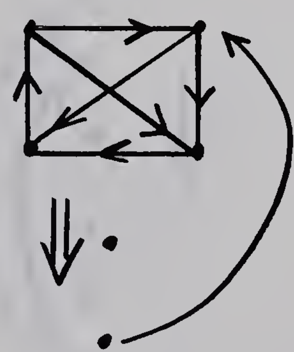
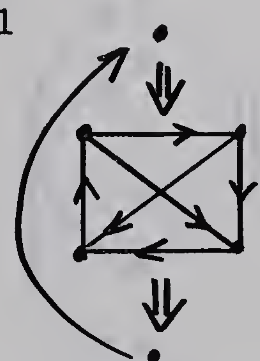
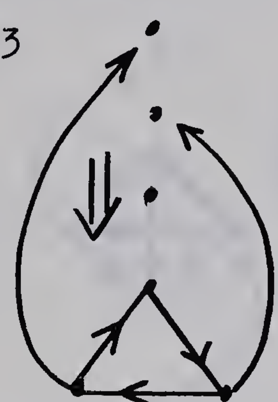
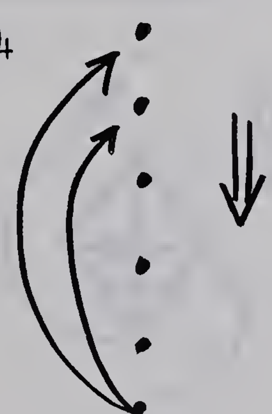
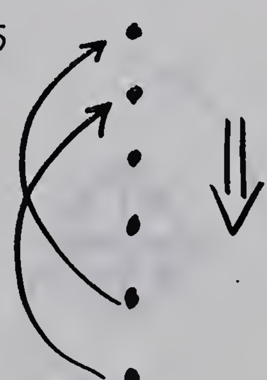
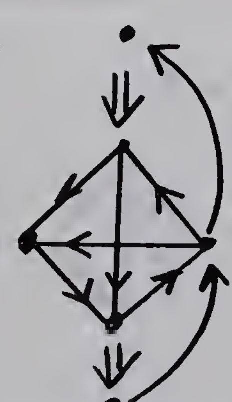
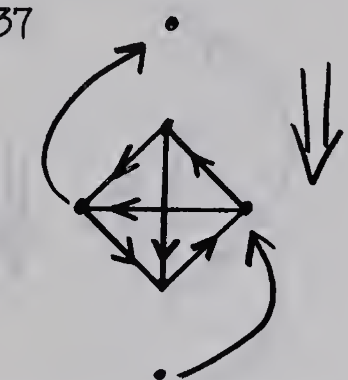
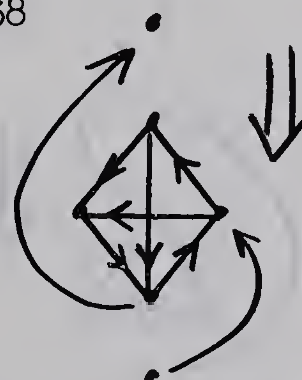
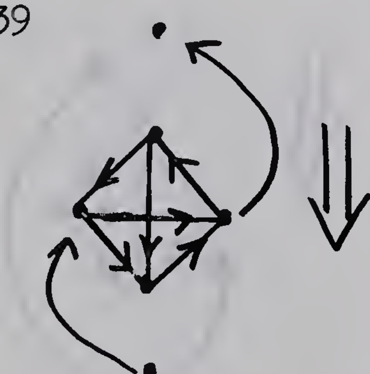
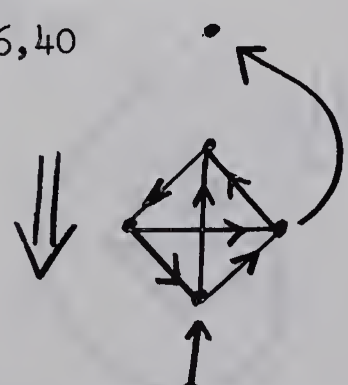
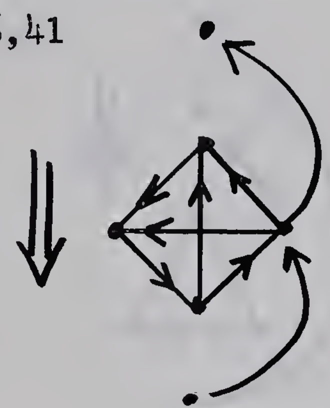
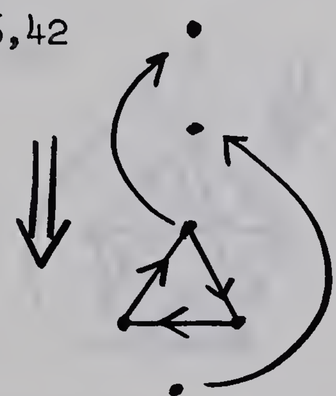
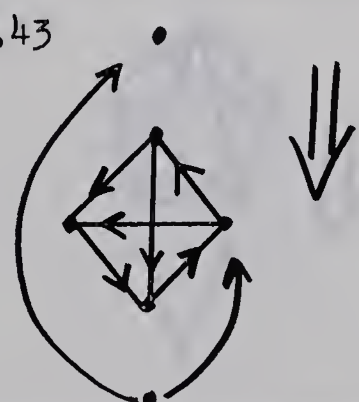
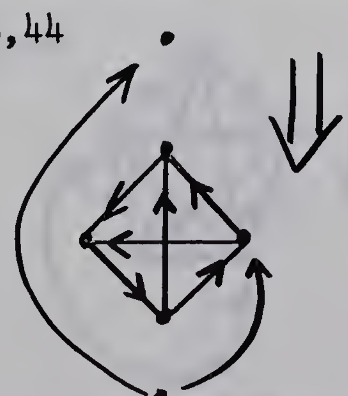
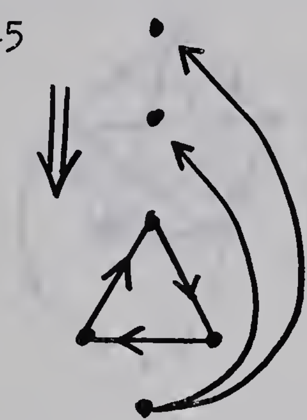
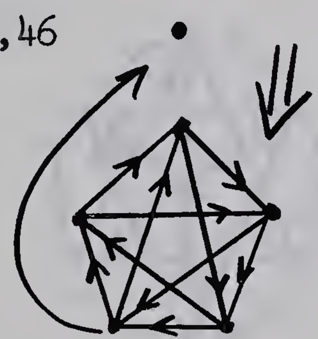
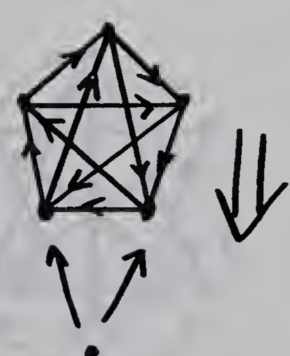
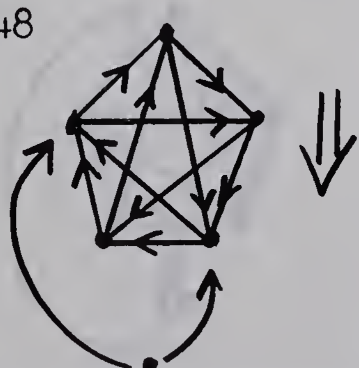
Score Vector	Tournament	Number of Labellings	Group	Remarks
(0,2,3,3,3,4)	$T_{6,25}$ 	720	I	Dual to $T_{6,9}$
(0,2,3,3,3,4)	$T_{6,26}$ 	720	I	Dual to $T_{6,10}$
(0,2,3,3,3,4)	$T_{6,27}$ 	240	C_3	Dual to $T_{6,11}$
(1,1,3,3,3,4)	$T_{6,28}$ 	240	C_3	Dual to $T_{6,22}$

Score Vector	Tournament	Number of Labellings	Group	Remarks
(1,1,3,3,3,4)	$T_{6,29}$ 	720	I	Dual to $T_{6,23}$
(1,1,3,3,3,4)	$T_{6,30}$ 	720	I	Dual to $T_{6,24}$
(1,2,2,3,3,4)	$T_{6,31}$ 	720	I	
(1,2,2,3,3,4)	$T_{6,32}$ 	720	I	

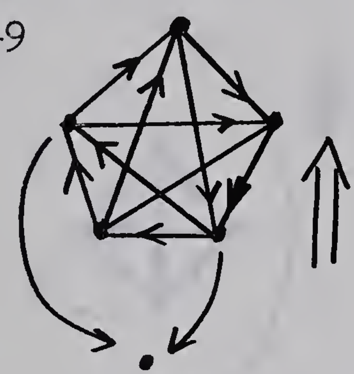
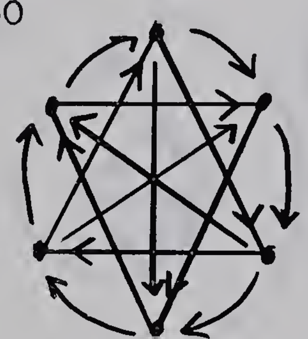
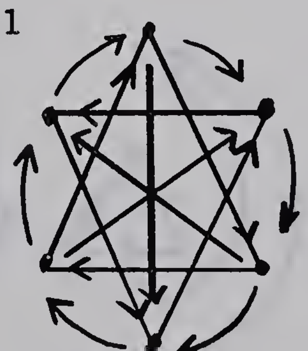
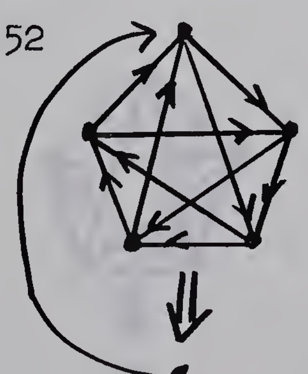
Score Vector	Tournament	Number of Labellings	Group	Remarks
(1,2,2,3,3,4)	$T_{6,33}$ 	720	I	
(1,2,2,3,3,4)	$T_{6,34}$ 	720	I	
(1,2,2,3,3,4)	$T_{6,35}$ 	720	I	
(1,2,2,3,3,4)	$T_{6,36}$ 	720	I	

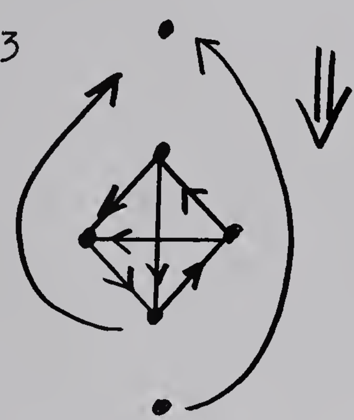
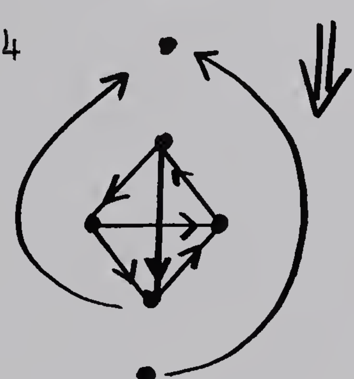
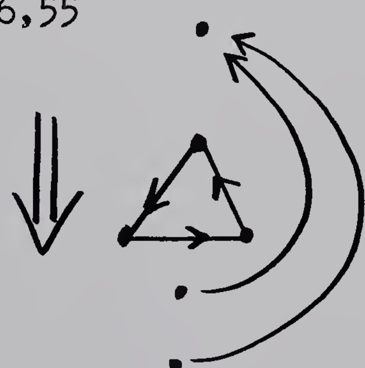
Score Vector	Tournament	Number of Labellings	Group	Remarks
(1,2,2,3,3,4)	$T_{6,37}$ 	720	I	
(1,2,2,3,3,4)	$T_{6,38}$ 	720	I	
(1,2,2,3,3,4)	$T_{6,39}$ 	720	I	
(1,2,2,3,3,4)	$T_{6,40}$ 	720	I	

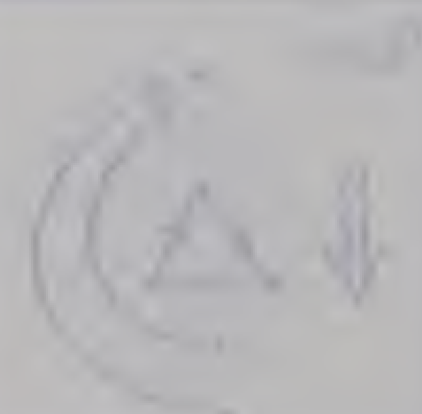
Score Vector	Tournament	Number of Labellings	Group	Remarks
(1,2,2,3,3,4)	$T_{6,41}$ 	720	I	
(1,2,2,3,3,4)	$T_{6,42}$ 	720	I	
(2,2,2,2,3,4)	$T_{6,43}$ 	720	I	
(2,2,2,2,3,4)	$T_{6,44}$ 	720	I	

Score Vector	Tournament	Number of Labellings	Group	Remarks
(2,2,2,2,3,4)	$T_{6,45}$ 	240	C_3	
(2,2,2,2,3,4)	$T_{6,46}$ 	720	I	
(2,2,2,3,3,3)	$T_{6,47}$ 	720	I	
(2,2,2,3,3,3)	$T_{6,48}$ 	720	I	

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
	[illegible]	[illegible]		[illegible]
	[illegible]	[illegible]		[illegible]
	[illegible]	[illegible]		[illegible]
	[illegible]	[illegible]		[illegible]

Score Vector	Tournament	Number of Labellings	Group	Remarks
(2,2,2,3,3,3)	$T_{6,49}$ 	720	I	
(2,2,2,3,3,3)	$T_{6,50}$ 	240	C_3	Nodes permute in tandem
(2,2,2,3,3,3)	$T_{6,51}$ 	240	C_3	Nodes permute in tandem
(1,2,3,3,3,3)	$T_{6,52}$ 	720	I	Dual to $T_{6,46}$

Score Vector	Tournament	Number of Labellings	Group	Remarks
(1,2,3,3,3,3)	$T_{6,53}$ 	720	I	Dual to $T_{6,43}$
(1,2,3,3,3,3)	$T_{6,54}$ 	720	I	Dual to $T_{6,44}$
(1,2,3,3,3,3)	$T_{6,55}$ 	240	C_3	Dual to $T_{6,45}$
(0,3,3,3,3,3)	$T_{6,56}$ 	144	C_5	Dual to $T_{6,12}$

[illegible]	[illegible]	[illegible]	[illegible]	[illegible]
[illegible]	[illegible]	[illegible]		[illegible]
[illegible]	[illegible]	[illegible]		[illegible]
[illegible]	[illegible]	[illegible]		[illegible]
[illegible]	[illegible]	[illegible]		[illegible]